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Vector refinement equations with infinitely supported masks☆

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Abstract

In this paper we investigate the L_2-solutions of vector refinement equations with exponentially decaying masks and a general dilation matrix. A vector refinement equation with a general dilation matrix and exponentially decaying masks is of the form

$$\phi(x) = \sum_{\alpha \in \mathbb{Z}^s} a(\alpha)\phi(Mx - \alpha), \quad x \in \mathbb{R}^s,$$

where the vector of functions $\phi = (\phi_1, \ldots, \phi_r)^T$ is in $(L_2(\mathbb{R}^s))^r$, $a =: (a(\alpha))_{\alpha \in \mathbb{Z}^s}$ is an exponentially decaying sequence of $r \times r$ matrices called refinement mask and M is an $s \times s$ integer matrix such that $\lim_{n\to\infty} M^{-n} = 0$. Associated with the mask a and dilation matrix M is a linear operator Q_a on $(L_2(\mathbb{R}^s))^r$ given by

$$Q_a f(x) := \sum_{\alpha \in \mathbb{Z}^s} a(\alpha) f(Mx - \alpha), \quad x \in \mathbb{R}^s, \quad f = (f_1, \ldots, f_r)^T \in (L_2(\mathbb{R}^s))^r.$$

The iterative scheme $(Q_a^n f)_{n=1,2,\ldots,}$ is called vector subdivision scheme or vector cascade algorithm. The purpose of this paper is to provide a necessary and sufficient condition to guarantee the sequence $(Q_a^n f)_{n=1,2,\ldots}$ to converge in L_2-norm. As an application, we also characterize biorthogonal multiple refinable functions, which extends some main results in [B. Han, R.Q. Jia, Characterization of Riesz bases

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of wavelets generated from multiresolution analysis, Appl. Comput. Harmon. Anal., to appear] and [R.Q. Jia, Convergence of vector subdivision schemes and construction of biorthogonal multiple wavelets, Advances in Wavelet (Hong Kong, 1997), Springer, Singapore, 1998, pp. 199–227] to the general setting.

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1. Introduction

A homogeneous vector refinement equation with mask a and a general dilation matrix M is the functional equation of the form

$$\phi(x) = \sum_{\alpha\in\mathbb{Z}^s} a(\alpha)\phi(Mx-\alpha), \quad x \in \mathbb{R}^s, \tag{1.1}$$

where $\phi = (\phi_1, \ldots, \phi_r)^T$ is unknown, a is an infinitely supported refinement mask such that each $a(\alpha)$ is an $r\times r$ complex number matrix and M is an $s\times s$ integer matrix such that $\lim_{n\to\infty} M^{-n}=0$. The solution of (1.1) is called multiple refinable functions. It is well known that refinement equations play an important role in wavelet analysis and computer graphics (see [1,5,15,12,10,18,29]). Most useful wavelets in applications are generated from refinable functions. The convergence of subdivision schemes in some Banach spaces is an important issue in wavelet analysis. For example, subdivision schemes can be used to characterize the existence and smoothness of solutions of Eq. (1.1) and also can be used to characterize orthogonality of the shifts of solutions of Eq. (1.1) with mask a having finitely supported. When mask a is infinitely supported, subdivision scheme associated with (1.1) is also important in wavelets analysis. It was known that subdivision scheme with masks having exponentially decay was used to characterize Riesz bases generated from multiresolution analysis in [9,12,19].

Before proceeding further, we introduce some notations. Let $(L_2(\mathbb{R}^s))^r$ denote the linear space of all vectors $f = (f_1, \ldots, f_r)^T$ such that $\|f\|_2 < \infty$, where

$$\|f\|_2 := \left(\sum_{j=1}^{r}\int_{\mathbb{R}^s}|f_j|^2\,dx\right)^{1/2}.$$

By $(L_{2,c}(\mathbb{R}^s))^r$ we denote the linear space of all compactly supported vectors of functions on $(L_2(\mathbb{R}^s))^r$.

The Fourier transform of a vector of functions in $(L_1(\mathbb{R}^s))^r$ is defined by

$$\hat{f}(\xi) := \int_{\mathbb{R}^s} f(x)e^{-ix\cdot\xi}\,dx, \quad \xi \in \mathbb{R}^s,$$

where $(L_1(\mathbb{R}^s))^r$ denotes the space of all Lebesgue integrable vectors of functions on $\mathbb{R}^s$, $\xi\cdot x$ is the inner product of two vectors ξ and x in $L_2(\mathbb{R}^s)$. The Fourier transform can be naturally extended to functions in $(L_2(\mathbb{R}^s))^r$. If mask a is an absolute summable sequence of $r\times r$ matrices on $\mathbb{Z}^s$, taking Fourier transform of both sides of (1.1), we obtain

$$\hat{\varphi}(\xi) = H((M^T)^{-1}\xi)\varphi((M^T)^{-1}\xi), \quad \xi \in \mathbb{R}^s, \tag{1.2}$$

where M^T denotes the transpose of M, and

$$H(\xi) = \frac{1}{m} \sum_{\alpha \in \mathbb{Z}^s} a(\alpha) e^{-i\alpha \cdot \xi}, \quad \xi \in \mathbb{R}^s$$

with $m = |\det M|$. Evidently, $H(\xi)$ is 2π-periodic. If $\hat{\varphi}(0) \neq 0$, then $\hat{\varphi}(0)$ is an eigenvector of the matrix $H(0)$ corresponding to eigenvalue 1.

When mask a is finitely supported, let φ be a solution of refinement equation (1.1). Furthermore, suppose φ is a compactly supported vector of functions in $(L_2(\mathbb{R}^s))^r$, satisfies $\hat{\varphi}(0) \neq 0$ and span $\{\hat{\varphi}(2\pi\beta) : \beta \in \mathbb{Z}^s\} = \mathbb{C}^r$, it was pointed out in [31] that 1 is a simple eigenvalue of $H(0)$ and other eigenvalues of $H(0)$ are less than 1 in modulus. In this paper we assume that these conditions are satisfied. In such a case, it is reasonable to assume that matrix $H(0)$ has the following form:

$$H(0) = \begin{pmatrix} 1 & 0 \\ 0 & \Lambda \end{pmatrix} \quad \text{where } \lim_{n \to \infty} \Lambda^n = 0. \tag{1.3}$$

For $j = 1, \ldots, r$, we use e_j to denote the *jth* column of the $r \times r$ identity matrix. Obviously, $e_1^T H(0) = e_1^T$ and $H(0)e_1 = e_1$.

Let M be a fixed dilation matrix with $m = |\det M|$. Then the coset spaces $\mathbb{Z}^s / M\mathbb{Z}^s$ consists of m elements. Let $\gamma_k + M\mathbb{Z}^s$, $k = 0, 1, \ldots, m-1$ be the m distinct elements of $\mathbb{Z}^s / M\mathbb{Z}^s$ with $\gamma_0 = 0$. We denote $E = \{\gamma_k, k = 0, 1, \ldots, m-1\}$. Thus, each element $\alpha \in \mathbb{Z}^s$ can be uniquely represented as $\varepsilon + M\gamma$, where $\varepsilon \in E$ and $\gamma \in \mathbb{Z}^s$.

We say that a satisfies the basic sum rule if for $k = 0, 1, \ldots, m-1$,

$$e_1^T \sum_{\alpha \in \mathbb{Z}^s} a(M\alpha + \gamma_k) = e_1^T. \tag{1.4}$$

If c is a (complex-valued) summable sequence on $\mathbb{Z}^s$, then its Fourier series is defined by

$$\hat{c}(\xi) := \sum_{\alpha \in \mathbb{Z}^s} c(\alpha) e^{-i\alpha \cdot \xi}, \quad \xi \in \mathbb{R}^s.$$

Evidently, $\hat{c}$ is a 2π-periodic continuous function on $\mathbb{R}^s$. When c is finitely supported, $\hat{c}$ is a trigonometric polynomial. We call $\hat{c}$, the symbol of c. We also define the Fourier series for c to be vector sequences or matrix sequences in similar ways.

It is easily seen that a summable sequence of $r \times r$ matrices a satisfies the basic sum rule if and only if

$$e_1^T \hat{a}(2\pi(M^T)^{-1}\gamma_k) = 0, \quad k = 0, 1, \ldots, m-1. \tag{1.5}$$

Let $(\ell_1(\mathbb{Z}^s))^{r \times r}$ be the linear space of all sequences of $r \times r$ matrices such that its each element absolutely converges on $\mathbb{Z}^s$. Suppose $a \in (\ell_1(\mathbb{Z}^s))^{r \times r}$ and M is a general dilation matrix. In order to solve the refinement equation (1.1), we introduce the linear operator Q_a on $(L_2(\mathbb{R}^s))^r$ given by

$$Q_a \phi(x) = \sum_{\alpha \in \mathbb{Z}^s} a(\alpha) \phi(Mx - \alpha), \quad x \in \mathbb{R}^s, \quad \phi \in (L_2(\mathbb{R}^s))^r. \tag{1.6}$$

Let ϕ_0 be a vector of compactly supported functions in $(L_2(\mathbb{R}^s))^r$. We consider the iteration scheme $\phi_n := Q_a^n \phi_0$, $n = 1, 2, \ldots$. This iteration scheme is called the vector cascade algorithm or vector subdivision schemes associated with mask a and a general dilation matrix M. Subdivision scheme has been extensively studied for the case in which mask a is finitely supported. A vector $\phi = (\phi_1, \ldots, \phi_r)^T \in (L_{2,c}(\mathbb{R}^s))^r$ is said to satisfy Strang–Fix conditions of order 1 if

$$e_1^T \hat{\phi}(0) = 1 \quad \text{and} \quad e_1^T \hat{\phi}(2\pi\beta) = 0 \quad \forall \beta \in \mathbb{Z}^s \backslash \{0\}. \tag{1.7}$$

By using the Poisson summation formula, it is easily seen that (1.7) is equivalent to the following condition:

$$e_1^T \sum_{\alpha \in \mathbb{Z}^s} \phi(\cdot - \alpha) = 1. \tag{1.8}$$

When mask a is finitely supported, in order for the subdivision scheme to converge in $(L_2(\mathbb{R}^s))^r$, initial vector of functions ϕ_0 must satisfy Strang–Fix conditions of order 1 [28].

When mask $a \in (\ell_1(\mathbb{Z}^s))^{r\times r}$, we say that the (vector) subdivision scheme associated with mask a and a general dilation matrix M converges in the L_2-norm, if there exists a vector $\phi \in (L_2(\mathbb{R}^s))^r$ such that for any $\phi_0 \in (L_{2,c}(\mathbb{R}^s))^r$ satisfying the Strang–Fix conditions of order 1,

$$\lim_{n\to\infty} \| Q_a^n \phi_0 - \phi \|_2 = 0.$$

If this is the case, then ϕ is a solution of the refinement equation (1.1) in $(L_2(\mathbb{R}^s))^r$.

Great efforts have been spent on the convergence of subdivision schemes mentioned above when masks a are finitely supported (see [1,10,11,21,23,27,28,33,35,36]).

Let $\phi = (\phi_1, \ldots, \phi_r)^T$ and $\tilde{\phi} = (\tilde{\phi}_1, \ldots, \tilde{\phi}_r)^T$ belong to $(L_2(\mathbb{R}^s))^r$. We say that the shifts of $\phi_1, \ldots, \phi_r$ and the shifts of $\tilde{\phi}_1, \ldots, \tilde{\phi}_r$ are biorthogonal, if

$$\langle \phi_j(\cdot - \alpha), \tilde{\phi}_k(\cdot - \beta)\rangle = \delta_{jk}\delta_{\alpha\beta} \quad \forall j, k = 1, \ldots, r, \quad \alpha, \beta \in \mathbb{Z}^s, \tag{1.9}$$

where δ_{jk} and $\delta_{\alpha\beta}$ stand for the Kronecker sign and $\langle \phi_j, \tilde{\phi}_k\rangle$ denotes the inner product of two functions ϕ_j and $\tilde{\phi}_k$ in $L_2(\mathbb{R}^s)$. If this is the case, then $\tilde{\phi}$ is said to be a dual to ϕ in $L_2(\mathbb{R}^s)$. If, in addition, ϕ and $\tilde{\phi}$ are multiple refinable functions, then ϕ and $\tilde{\phi}$ are a pair of biorthogonal multiple refinable functions.

In engineering, infinitely supported masks are needed [9]. Due to some desirable properties, infinitely supported masks with exponentially decaying and fractional splines [37] are of interest in the area of digital signal processing in electrical engineering [3,4,6,16,32]. To study Riesz bases of wavelets generated from multiresolution analysis, the L_2-convergence of subdivision schemes with mask a having exponential decay was investigated for $r = 1$ by Han and Jia [12], for $s = 1$ and $M = 2$ by Jia [19]. Han [9] also characterized the convergence of subdivision scheme with mask a having exponential decay in weighted subspaces of $L_2(\mathbb{R})$ when $r = 1$, $s = 1$ and $M = 2$. As pointed out in [9] that the study of subdivision schemes and refinable functions with infinitely supported masks is not a trivial generalization of known results in the literature mentioned above. To study refinement equation, the spectral theory of compact operator is involved for the case in which mask a is infinitely supported.

The purpose of this paper is to investigate the vector refinement equation with mask a having exponential decay and a general dilation matrix M. In this paper, we will characterize the L_2-solutions of refinement equations (1.1). A necessary and sufficient condition for the convergence of subdivision schemes with this mask a and a general dilation matrix M in $(L_2(\mathbb{R}^s))^r$ is obtained. As an application, we also characterize the biorthogonal multiple refinable functions, which extends some main results in [11,12,19,20] to the general setting.

2. Notations and lemmas

Let E_μ denote linear space of all sequences u on $\mathbb{Z}^s$ for which

$$\|u\|_{E_\mu} := \sum_{\alpha\in\mathbb{Z}^s} |u(\alpha)|e^{\mu|\alpha|} < \infty,$$

where the vector norm $|\cdot|$ on $\mathbb{R}^s$ so chosen that $\|M^{-1}\| := \sup_{|x|\leqslant 1} |M^{-1}x| < 1$. Equipped with the norm $\|\cdot\|_{E_\mu}$, E_μ becomes a Banach space. Let E_μ^r denote the linear space of all vectors of sequences $u = (u_1, \ldots, u_r)^T$ such that $u_1, \ldots, u_r \in E_\mu$. The norm on E_μ^r is defined by

$$\|u\|_{E_\mu^r} := \sum_{j=1}^{r} \|u_j\|_{E_\mu}.$$

By $E_\mu^{r\times r}$ we denote the linear space of all matrices of sequences $u(\alpha) = (u_{j,k}(\alpha))_{1\leqslant j,k\leqslant r}$ such that $u_{j,k} \in E_\mu$ $j, k = 1, \ldots, r$. The norm on $E_\mu^{r\times r}$ is defined by

$$\|u\|_{E_\mu^{r\times r}} := \sum_{j=1}^{r}\sum_{k=1}^{r} \|u_{j,k}\|_{E_\mu}.$$

We point out that the spaces E_μ, E_μ^r and $E_\mu^{r\times r}$ were used by Cohen and Daubechies [4], by Jia [19] and by Han and Jia [12]. When $\mu = 0$, E_0, E_0^r and $E_0^{r\times r}$ are the usual $\ell_1(\mathbb{Z}^s)$, $(\ell_1(\mathbb{Z}^s))^r$ and $(\ell_1(\mathbb{Z}^s))^{r\times r}$ spaces.

Before going on, we introduce the Kronecker product of two matrices. The Kronecker product of two matrices is an important tool in the study of vector refinement equations (see [19,21,24,25,38]). Let us mention some useful properties of the Kronecker product from [26,7]. Let $A = (a_{i,j})_{1\leqslant i\leqslant m, 1\leqslant j\leqslant n}$, and $B = (b_{i,j})_{1\leqslant i\leqslant k, 1\leqslant j\leqslant l}$, be two matrices. The (right) Kronecker product of A and B, written $A \otimes B$, is defined to be the block matrix

$$A \otimes B := \begin{pmatrix} a_{11}B & a_{12}B & \cdots & a_{1n}B \\ a_{21}B & a_{22}B & \cdots & a_{2n}B \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1}B & a_{m2}B & \cdots & a_{mn}B \end{pmatrix}.$$

For three matrices A, B and C of the same type, we have

$$(A + B) \otimes C = (A \otimes C) + (B \otimes C),$$

$$A \otimes (B + C) = (A \otimes B) + (A \otimes C).$$

If A, B, C, D are four matrices such that the products AC and BD are well defined, then

$$(A \otimes B)(C \otimes D) = (AC) \otimes (BD).$$

If $\lambda_1, \ldots, \lambda_r$ are the eigenvalues of an $r \times r$ matrix A and $\mu_1, \ldots, \mu_r$ are the eigenvalues of an $r \times r$ matrix B, it follows from [26] that the eigenvalues of $A \otimes B$ are $\lambda_j\mu_k$, $j, k = 1, \ldots, r$.

For two functions f, g in $L_2(\mathbb{R}^s)$, $f \odot g$ is defined as follows:

$$f \odot g(x) := \int_{\mathbb{R}^s} f(x + y)\overline{g(y)}\, dy, \quad x \in \mathbb{R}^s,$$

where $\overline{g(y)}$ stands for the complex conjugate of $g(y)$. In other words, $f \odot g$ is the convolution of f with the function $y \mapsto \overline{g(-y)}$, $y \in \mathbb{R}^s$. It is easily seen that $f \odot g$ lies in $C_0(\mathbb{R}^s)$, the space of continuous functions on $\mathbb{R}^s$ which vanish at ∞. Evidently

$$\|f \odot g\|_\infty \leqslant \|f\|_2 \|g\|_2. \tag{2.1}$$

Moreover $(f \odot f)(0) = \|f\|_2^2$.

For a matrix $A = (a_{ij})_{1 \leqslant i,j \leqslant r}$, the vector

$$(a_{11}, \ldots, a_{r1}, a_{12}, \ldots, a_{r2}, \ldots, a_{1r}, \ldots, a_{rr})^T$$

is said to be the vec-function of A and written as $vec\, A$. Suppose A, X and B are three $r \times r$ matrices. Then we have (see [17])

$$vec(AXB) = (B^T \otimes A) vec\, X. \tag{2.2}$$

Suppose $\phi = (\phi_1, \ldots, \phi_r)^T$ and $\psi = (\psi_1, \ldots, \psi_r)^T$ belong to $(L_2(\mathbb{R}^s))^r$, let $\phi \odot \psi^T$ be defined as follows:

$$\phi \odot \psi^T := \begin{pmatrix} \phi_1 \odot \psi_1 & \phi_1 \odot \psi_2 & \cdots & \phi_1 \odot \psi_r \\ \phi_2 \odot \psi_1 & \phi_2 \odot \psi_2 & \cdots & \phi_2 \odot \psi_r \\ \vdots & \vdots & \ddots & \vdots \\ \phi_r \odot \psi_1 & \phi_r \odot \psi_2 & \cdots & \phi_r \odot \psi_r \end{pmatrix}.$$

By (2.1) we have

$$\|vec(\phi \odot \psi^T)\|_\infty \leqslant \|\phi\|_2 \|\psi\|_2 \tag{2.3}$$

and

$$|vec(\phi \odot \phi^T)(0)| = \sum_{j=1}^r \sum_{k=1}^r |\phi_j \odot \phi_k(0)| \geqslant \sum_{j=1}^r |\phi_j \odot \phi_j(0)| = \sum_{j=1}^r \|\phi_j\|_2^2.$$

Consequently

$$|vec(\phi \odot \phi^T)(0)| \geqslant \|\phi\|_2^2. \tag{2.4}$$

Suppose mask $a \in E_\mu^{r \times r}$ for some $\mu > 0$, let b be defined as follows:

$$b(\alpha) := \sum_{\beta \in \mathbb{Z}^s} \overline{a(\beta)} \otimes a(\alpha + \beta)/m, \quad \alpha \in \mathbb{Z}^s. \tag{2.5}$$

It follows from a simple computation that b lies in $E_\mu^{r^2 \times r^2}$. By (2.5), we have

$$\sum_{\alpha \in \mathbb{Z}^s} b(\alpha)/m = \left(\sum_{\beta \in \mathbb{Z}^s} \overline{a(\beta)}/m \right) \otimes \left(\sum_{\alpha \in \mathbb{Z}^s} a(\alpha + \beta)/m \right) = \overline{H(0)} \otimes H(0)$$

and

$$(\overline{e_1^T} \otimes e_1^T)(\overline{H(0)} \otimes H(0)) = (\overline{e_1^T H(0)}) \otimes (e_1^T H(0)) = \overline{e_1^T} \otimes e_1^T.$$

It follows from above discussions that the matrix $\sum_{\alpha\in\mathbb{Z}^s} b(\alpha)/m$ has a simple eigenvalue 1, $\overline{e_1^T}\otimes e_1^T$ is a left eigenvector of $\sum_{\alpha\in\mathbb{Z}^s} b(\alpha)/m$ corresponding to eigenvalue 1, and other eigenvalues of $\sum_{\alpha\in\mathbb{Z}^s} b(\alpha)/m$ are less than 1 in modulus. If a satisfies the basic sum rule, we claim that b also satisfies the basic sum rule. In fact, for $\gamma_k \in E$, $k = 0, 1, \ldots, m-1$,

$$\begin{aligned}(\overline{e_1^T}\otimes e_1^T)\sum_{\alpha\in\mathbb{Z}^s} b(M\alpha+\gamma_k) &= (\overline{e_1^T}\otimes e_1^T)\sum_{\alpha\in\mathbb{Z}^s}\sum_{\beta\in\mathbb{Z}^s}\overline{a(\beta)}\otimes a(M\alpha+\gamma_k+\beta)/m\\ &= \overline{e_1^T\sum_{\beta\in\mathbb{Z}^s} a(\beta)}\otimes e_1^T\sum_{\alpha\in\mathbb{Z}^s} a(M\alpha+\gamma_k+\beta)/m\\ &= \overline{e_1^T\sum_{\beta\in\mathbb{Z}} a(\beta)}\otimes e_1^T/m = \overline{e_1^T}\otimes e_1^T.\end{aligned}$$

It is easily seen that b satisfies the basic sum rule if and only if $(\overline{e_1^T}\otimes e_1^T)\hat{b}(2\pi(M^T)^{-1}\gamma_k) = 0$, $k = 0, 1, \ldots, m-1$. By the definition of b, we know that $(\overline{e_1^T}\otimes e_1^T)\hat{b}(2\pi(M^T)^{-1}\gamma_k) = 0$ imply $e_1^T\hat{a}(2\pi(M^T)^{-1}\gamma_k) = 0$, for $k = 0, 1, \ldots, m-1$. Hence a satisfies the basic sum rule. Above discussions tell us that a satisfies the basic sum rule if and only if b also satisfies the basic sum rule.

For $a \in E_\mu^{r\times r}$, let T_a be the transition operator on E_μ^r defined by

$$T_a u(\alpha) := \sum_{\beta\in\mathbb{Z}^s} a(M\alpha-\beta)u(\beta), \quad \alpha\in\mathbb{Z}^s,\ \ u\in E_\mu^r. \tag{2.6}$$

See [1,8,10,12,18–21,23–25] for some earlier works on this operator.

It follows from the proofs of Lemmas 3.1 and 3.2 in [12] that the transition operator T_a is a bounded and compact operator on E_μ^r.

Lemma 2.1. *Let $a \in E_\mu^{r\times r}$ for some $\mu > 0$. Then the transition operator T_a is a bounded operator on E_μ^r. Moreover,*

$$\|T_a u\|_{E_\mu^r} \leqslant \|a\|_{E_\mu^{r\times r}}\|u\|_{E_\mu^r} \quad \forall u \in E_\mu^r. \tag{2.7}$$

Proof. For $\alpha \in \mathbb{Z}^s$, we have

$$|\alpha| = |M^{-1}M\alpha| \leqslant \|M^{-1}\||M\alpha| < |M\alpha-\beta| + |\beta|.$$

Consequently

$$\|T_a u\|_{E_\mu^r} \leqslant \sum_{\beta\in\mathbb{Z}^s}\left(\sum_{\alpha\in\mathbb{Z}^s}|a(M\alpha-\beta)u(\beta)|e^{\mu|M\alpha-\beta|}\right)e^{\mu|\beta|} \leqslant \|a\|_{E_\mu^{r\times r}}\|u\|_{E_\mu^r}.$$

Hence, T_a is a bounded operator on E_μ^r. $\square$

Lemma 2.2. *Let $a \in E_\mu^{r\times r}$ for some $\mu > 0$. Then the transition operator T_a is a compact operator on E_μ^r.*

Proof. When a is finitely supported, then T_a is the limit of a sequence of finite-rank operator. Hence T_a is a compact operator. In general, for $L = 1, 2, \ldots,$ let a_L be the sequences in $E_\mu^{r\times r}$

defined by $a_L(\alpha) = a(\alpha)$ for $|\alpha| \leqslant L$, and $a_L(\alpha) = 0$ for $|\alpha| > L$. Each a_L is finitely supported, then each T_{a_L} is a compact operator for $L = 1, 2, \ldots$. By the definition of a_L, we obtain

$$\begin{aligned}\sum_{\alpha\in\mathbb{Z}^s} |(T_{a_L} - T_a)u(\alpha)|e^{\mu|\alpha|} &= \sum_{\alpha\in\mathbb{Z}^s} \sum_{|M\alpha-\beta|>L} |a(M\alpha-\beta)u(\beta)|e^{\mu|\alpha|} \\ &\leqslant \sum_{\alpha\in\mathbb{Z}^s} \sum_{|M\alpha-\beta|>L} |a(M\alpha-\beta)u(\beta)|e^{\mu|M\alpha-\beta|}e^{\mu|\beta|} \\ &\leqslant \|u\|_{E_\mu^r} \sum_{|\gamma|>L} \sum_{j=1}^{r}\sum_{k=1}^{r} |a_{jk}(\gamma)|e^{\mu|\gamma|},\end{aligned}$$

where $a(\alpha) = (a_{jk}(\alpha))_{1\leqslant j,k\leqslant r}$.

Therefore

$$\lim_{L\to\infty} \|T_{a_L} - T_a\| \leqslant \lim_{L\to\infty} \sum_{j=1}^{r}\sum_{k=1}^{r}\sum_{|\gamma|>L} |a_{jk}(\gamma)|e^{\mu|\gamma|} = 0.$$

It follows from the above estimate that T_a is a compact operator. □

Since T_a is a compact linear operator on $E_\mu^{r\times r}$, the Riesz Theory of compact operators (see Chapter 3 in [34]) says that the spectrum of T_a is a countable compact set whose only possible limit point is 0. In particular, there exists an eigenvalue σ of T_a such that $\rho(T_a) = |\sigma|$, where $\rho(T_a)$ denotes the spectral radius of T_a. It follows from [12] that if $(T_n)_{n=1,2,\ldots,}$ is a sequence of bounded linear operators on Banach space $E_\mu^{r\times r}$ such that $\|T_n - T\| \to 0$ as $n \to \infty$, then $\lim_{n\to\infty} \rho(T_n) = \rho(T)$.

Consider the subspace V of $E_\mu^{r^2}$ defined by

$$V := \left\{ v \in E_\mu^{r^2} : (\overline{e_1^T} \otimes e_1^T) \sum_{\alpha\in\mathbb{Z}^s} v(\alpha) = 0 \right\}. \tag{2.8}$$

Theorem 2.3. *Let $b \in E_\mu^{r^2\times r^2}$ be defined by* (2.5). *Then V is invariant under T_b, if and only if b satisfies the basic sum rule.*

Proof. Let b satisfy the basic sum rule and $v \in V$. Then we have

$$\begin{aligned}(\overline{e_1^T} \otimes e_1^T) \sum_{\alpha\in\mathbb{Z}^s} T_b v(\alpha) &= (\overline{e_1^T} \otimes e_1^T) \sum_{\alpha\in\mathbb{Z}^s}\sum_{\beta\in\mathbb{Z}^s} b(M\alpha-\beta)v(\beta) \\ &= \sum_{\beta\in\mathbb{Z}^s} \left(\sum_{\alpha\in\mathbb{Z}^s} (\overline{e_1^T} \otimes e_1^T) b(M\alpha-\beta) \right) v(\beta) \\ &= (\overline{e_1^T} \otimes e_1^T) \sum_{\beta\in\mathbb{Z}^s} v(\beta) = 0.\end{aligned}$$

Therefore $v \in V$ implies $T_b v \in V$. This proves that V is invariant under T_b.

Next, we prove the necessity part of the theorem. Note that for $\gamma_i \in E, i = 0, 1, \ldots,$ m-1, $(\overline{e_k} \otimes e_j)\nabla_{-\gamma_i}\delta \in V$, for $j, k = 1, 2, \ldots, r$, where for a vector $\lambda \in \mathbb{Z}^s$, the difference operator ∇_λ is defined by

$$\nabla_\lambda v := v - v(\cdot - \lambda), \quad v \in l(\mathbb{Z}^s),$$

and for $\beta \in \mathbb{Z}^s$, the sequence δ_β on $\mathbb{Z}^s$ given by

$$\delta_\beta(\alpha) = \begin{cases} 1 & \text{for } \alpha = \beta, \\ 0 & \text{for } \alpha \in \mathbb{Z}^s \backslash \{\beta\}. \end{cases}$$

If $\beta = 0$, we write δ for δ_0.

Hence

$$\sum_{\alpha\in\mathbb{Z}^s} (\overline{e_1^T} \otimes e_1^T) \left[b(M\alpha) - b(M\alpha + \gamma_i) \right] (\overline{e_k} \otimes e_j) = (\overline{e_1^T} \otimes e_1^T) \sum_{\alpha\in\mathbb{Z}^s} T_b(\overline{e_k} \otimes e_j \nabla_{-\gamma_i} \delta)(\alpha) = 0.$$

It follows that for $j, k = 1, 2, \ldots, r$,

$$(\overline{e_1^T} \otimes e_1^T) \sum_{\alpha\in\mathbb{Z}^s} b(M\alpha)(\overline{e_k} \otimes e_j) = (\overline{e_1^T} \otimes e_1^T) \sum_{\alpha\in\mathbb{Z}^s} b(M\alpha + \gamma_i)(\overline{e_k} \otimes e_j).$$

Since the above relation is true for all $j, k = 1, 2, \ldots, r$. Therefore

$$(\overline{e_1^T} \otimes e_1^T) \sum_{\alpha\in\mathbb{Z}^s} b(M\alpha) = (\overline{e_1^T} \otimes e_1^T) \sum_{\alpha\in\mathbb{Z}^s} b(M\alpha + \gamma_i).$$

Since $e_1^T \sum_{\alpha\in\mathbb{Z}^s} a(\alpha) = m e_1^T$. We have

$$\begin{aligned}(\overline{e_1^T} \otimes e_1^T) \sum_{\alpha\in\mathbb{Z}^s} b(\alpha) &= (\overline{e_1^T} \otimes e_1^T) \sum_{\alpha\in\mathbb{Z}^s} \sum_{\beta\in\mathbb{Z}^s} \overline{a(\beta)} \otimes a(\alpha + \beta)/m \\ &= \sum_{\beta\in\mathbb{Z}^s} \overline{e_1^T a(\beta)} \otimes \sum_{\alpha\in\mathbb{Z}^s} e_1^T a(\alpha + \beta)/m = \sum_{\beta\in\mathbb{Z}^s} \overline{e_1^T a(\beta)} \otimes e_1^T \\ &= m\overline{e_1^T} \otimes e_1^T.\end{aligned}$$

It follows that

$$(\overline{e_1^T} \otimes e_1^T) \sum_{\alpha\in\mathbb{Z}^s} b(M\alpha + \gamma_i) = \overline{e_1^T} \otimes e_1^T, \quad \gamma_i \in E, \ \ i = 0, 1, \ldots, m-1.$$

Hence b satisfies the basic sum rule. $\square$

Suppose $\phi \in (L_2(\mathbb{R}^s))^r$ is a solution of the refinement equation (1.1), where the mask a is assumed to be in $(l_1(\mathbb{Z}^s))^{r\times r}$ for the time being, then

$$\phi \odot \phi^T = \sum_{\alpha\in\mathbb{Z}^s} \sum_{\beta\in\mathbb{Z}^s} a(\alpha)\phi(M \cdot - \alpha) \odot \phi^T(M \cdot - \beta)\overline{a(\beta)}^T.$$

Since

$$\phi(M \cdot - \alpha) \odot \phi^T(M \cdot - \beta) = \frac{1}{m} \phi \odot \phi^T(M \cdot - \alpha + \beta).$$

By (2.2), we have

$$vec(a(\alpha)\phi(M \cdot - \alpha) \odot \phi^T(M \cdot - \beta)\overline{a(\beta)}^T) = \frac{1}{m}\overline{a(\beta)} \otimes a(\alpha) vec(\phi \odot \phi^T)(M \cdot - \alpha + \beta).$$

Hence

$$vec(\phi \odot \phi^T) = \sum_{\alpha\in\mathbb{Z}^s} \sum_{\beta\in\mathbb{Z}^s} \frac{1}{m}\overline{a(\beta)} \otimes a(\alpha) vec(\phi \odot \phi^T)(M \cdot -\alpha + \beta). \tag{2.9}$$

Let $f := vec(\phi \odot \phi^T)$, then $f \in (C_0(\mathbb{R}^s))^{r^2}$, the linear space of $r^2 \times 1$ vectors of functions in $C_0(\mathbb{R}^s)$. It is easily checked that f satisfies the refinement equation as follows:

$$f = \sum_{\alpha\in\mathbb{Z}^s} b(\alpha) f(M \cdot -\alpha),$$

where b is given by (2.5).

For $n = 1, 2, \ldots$, let $a_1 = a$ and a_n be defined by the following iterative relations:

$$a_n(\alpha) = \sum_{\beta\in\mathbb{Z}^s} a_{n-1}(\beta) a(\alpha - M\beta), \quad \alpha \in \mathbb{Z}^s. \tag{2.10}$$

By (1.2) and induction on n, it is easily seen that

$$Q_a^n \phi = \sum_{\alpha\in\mathbb{Z}^s} a_n(\alpha) \phi(M^n \cdot -\alpha). \tag{2.11}$$

Similarly, for $f \in (C_0(\mathbb{R}^s))^{r^2}$, we have

$$Q_b^n f = \sum_{\alpha\in\mathbb{Z}^s} b_n(\alpha) f(M^n \cdot -\alpha), \tag{2.12}$$

where b_n $(n = 1, 2, \ldots)$ are the sequences of $r^2 \times r^2$ matrices defined as follows:

$$b_1 = b \quad \text{and} \quad b_n(\alpha) = \sum_{\beta\in\mathbb{Z}^s} b_{n-1}(\beta) b(\alpha - M\beta), \quad \alpha \in \mathbb{Z}^s. \tag{2.13}$$

It was proved in [21] that a_n and b_n satisfy the following relations:

$$b_n(\alpha) = \sum_{\beta\in\mathbb{Z}^s} \overline{a_n(\beta)} \otimes a_n(\alpha + \beta)/m^n, \quad \alpha \in \mathbb{Z}^s, \; n = 1, 2, \ldots . \tag{2.14}$$

The following is an outline of this proof. We can prove (2.14) by induction on n. By the definition of b, (2.14) holds true for $n = 1$. Suppose $n > 1$ and (2.14) is valid for $n - 1$. For $\alpha \in \mathbb{Z}^s$, we have

$$\begin{aligned} b_n(\alpha) &= \sum_{\eta\in\mathbb{Z}^s} b_{n-1}(\eta) b(\alpha - M\eta) \\ &= m^{-n} \sum_{\eta\in\mathbb{Z}^s} \left(\sum_{\gamma\in\mathbb{Z}^s} \overline{a_{n-1}(\gamma)} \otimes a_{n-1}(\eta + \gamma) \right) \left(\sum_{\beta\in\mathbb{Z}^s} \overline{a(\beta)} \otimes a(\alpha - M\eta + \beta) \right) \\ &= m^{-n} \sum_{\beta\in\mathbb{Z}^s} \sum_{\gamma\in\mathbb{Z}^s} \sum_{\eta\in\mathbb{Z}^s} \overline{a_{n-1}(\gamma) a(\beta - M\gamma)} \otimes (a_{n-1}(\eta) a(\alpha + \beta - M\eta)) \\ &= m^{-n} \sum_{\beta\in\mathbb{Z}^s} \overline{a_n(\beta)} \otimes a_n(\alpha + \beta), \end{aligned}$$

which implies that (2.14) is true for all n.

Let ϕ_0 and ψ_0 lie in $(L_2(\mathbb{R}^s))^r$. It follows from above discussions that

$$vec((Q_a^n\phi_0)\odot(Q_a^n\psi_0)^T)=\sum_{\alpha\in\mathbb{Z}^s}\sum_{\beta\in\mathbb{Z}^s}m^{-n}\overline{a_n(\beta)}\otimes a_n(\alpha)vec(\phi_0\odot\psi_0^T)(M^n\cdot-\alpha+\beta).$$

By (2.14), we have, for $n=1,2,\ldots,$

$$vec((Q_a^n\phi_0)\odot(Q_a^n\psi_0)^T)=Q_b^n(vec(\phi_0\odot\psi_0^T)). \tag{2.15}$$

Theorem 2.4. *Suppose $a\in E_\mu^{r\times r}$ for some $\mu>0$, $H(0)=\sum_{\alpha\in\mathbb{Z}^s}a(\alpha)/m$ satisfies* (1.3). *Let b and T_b be given by* (2.5) *and* (2.6), *respectively, then $\rho(T_b|_{E_\mu^{r^2}})\geqslant 1$.*

Proof. First, we consider the case when a is finitely supported. Let ϕ_0 be the characteristic function of the unit cube $[0,1]^s$. By (2.4) we have

$$\|Q_a^n(e_1\phi_0)\|_2^2\leqslant\left|vec((Q_a^n(e_1\phi_0))\odot(Q_a^n(e_1\phi_0))^T)(0)\right|=\left|Q_b^n(vec((e_1\phi_0)\odot(e_1\phi_0)^T))(0)\right|.$$

By an induction on n, we have

$$T_b^n v(\alpha)=\sum_{\beta\in\mathbb{Z}^s}b_n(M^n\alpha-\beta)v(\beta). \tag{2.16}$$

It follows that

$$\|Q_a^n(e_1\phi_0)\|_2^2\leqslant\left|Q_b^n(vec((e_1\phi_0)\odot(e_1\phi_0)^T))(0)\right|=\left|T_b^n(vec((e_1\phi_0)\odot(e_1\phi_0)^T))(0)\right|.$$

If $\rho(T_b|_{E_\mu^{r^2}})<1$, then $Q_a^n(e_1\phi_0)$ would converge to 0 in the L_2-norm, as $n\to\infty$. Since $H(0)=\sum_{\alpha\in\mathbb{Z}^s}a(\alpha)/m$ satisfies (1.3), by a simple computation, we have

$$\hat{Q}_a^n(e_1\phi_0)(0)=H(0)^n e_1=e_1.$$

This contradiction demonstrates that $\rho(T_b|_{E_\mu^{r^2}})\geqslant 1$.

For the general cases, suppose $a\in E_\mu^{r\times r}$ for some $\mu>0$. For $L=1,2,\ldots,$ we can find matrix sequences a_L $(L=1,2,\ldots)$ such that each a_L is supported on $[-L,L]^s$, $e_1^T\sum_{\alpha\in\mathbb{Z}}a_L(\alpha)/m=e_1^T$ and $\|a_L-a\|_{E_\mu^{r\times r}}\to 0$ as $L\to\infty$. Let $b_L(\alpha)=\sum_{\beta\in\mathbb{Z}^s}\overline{a_L(\beta)}\otimes a_L(\alpha+\beta)/m$, by Lemma 2.1, $\|T_{b_L}|_{E_\mu^{r^2}}-T_b|_{E_\mu^{r^2}}\|\to 0$ as $L\to\infty$. It follows that $\lim_{L\to\infty}\rho(T_{b_L}|_{E_\mu^{r^2}})=\rho(T_b|_{E_\mu^{r^2}})$. If $\rho(T_b|_{E_\mu^{r^2}})<1$, then $\rho(T_{b_L}|_{E_\mu^{r^2}})<1$ for sufficiently large L, which is impossible. Therefore, we have $\rho(T_b|_{E_\mu^{r^2}})\geqslant 1$. □

3. Convergence of subdivision scheme

In this section, we will show that the vector subdivision schemes associated with a having exponentially decay and a general dilation matrix M converges in the L_2-norm if and only if V is invariant under T_b and $\rho(T_b|_V)<1$. Our proofs are based on [12,19,23].

Theorem 3.1. *Let $a\in E_\mu^{r\times r}$ for some $\mu>0$ and $H(0)=\sum_{\alpha\in\mathbb{Z}^s}a(\alpha)/m$ satisfies* (1.3). *Suppose b is given by* (2.5) *and T_b is defined by* (2.6). *Then the subdivision scheme associated with mask* *a and a general dilation matrix M converges in the L_2-norm if and only if*

(1) $\lim_{n\to\infty} \|T_b^n v\|_\infty = 0,\ \forall v \in V$.
(2) *a satisfies the basic sum rule*,

where V is defined by (2.8).

Proof. We first establish the necessity part of the theorem. We choose χ to be the characteristic function of the unit interval [0, 1). Then $e_1\chi$ satisfies the moment conditions of order 1, and $vec((e_1\chi) \odot (e_1\chi)^T) = (\overline{e_1} \otimes e_1)h$, where h is the hat function given by $h(x) := \max\{1 - |x|, 0\}$, $x \in \mathbb{R}$. We have that $Q_a^n(e_1\chi)$ converges to some limit function ϕ in the L_2-norm.

By (2.3), we have

$$\begin{aligned}&\left\|vec(Q_a^n(e_1\chi) \odot (Q_a^n(e_1\chi))^T - \phi \odot \phi^T)\right\|_\infty \\ &\quad\leqslant \left\|vec(Q_a^n(e_1\chi) \odot (Q_a^n(e_1\chi) - \phi)^T)\right\|_\infty + \left\|vec((Q_a^n(e_1\chi) - \phi) \odot \phi^T)\right\|_\infty \\ &\quad\leqslant \left\|Q_a^n(e_1\chi)\right\|_2 \left\|Q_a^n(e_1\chi) - \phi\right\|_2 + \left\|Q_a^n(e_1\chi) - \phi\right\|_2 \|\phi\|_2 .\end{aligned}$$

Which implies that $vec(Q_a^n(e_1\chi) \odot (Q_a^n(e_1\chi))^T)$ converges to $vec(\phi \odot \phi^T)$ uniformly. By (2.15), we have $Q_b^n(\overline{e_1} \otimes e_1h)$ converges to $vec(\phi \odot \phi^T)$ uniformly. Since $vec(\phi \odot \phi^T)$ is uniformly continuous, and

$$\begin{aligned}&\left\|Q_b^n(\overline{e_1} \otimes e_1h) - Q_b^n(\overline{e_1} \otimes e_1h)(\cdot - M^{-n}e_j)\right\|_\infty \\ &\quad\leqslant \left\|Q_b^n(\overline{e_1} \otimes e_1h) - vec(\phi \odot \phi^T)\right\|_\infty + \left\|vec(\phi \odot \phi^T) - vec(\phi \odot \phi^T)(\cdot - M^{-n}e_j)\right\|_\infty \\ &\qquad + \left\|Q_b^n(\overline{e_1} \otimes e_1h)(\cdot - M^{-n}e_j) - vec(\phi \odot \phi^T)(\cdot - M^{-n}e_j)\right\|_\infty .\end{aligned}$$

Consequently,

$$\lim_{n\to\infty} \left\|Q_b^n(\overline{e_1} \otimes e_1h) - Q_b^n(\overline{e_1} \otimes e_1h)(\cdot - M^{-n}e_j)\right\|_\infty = 0.$$

It follows from (2.12) that

$$Q_b^n(\overline{e_1} \otimes e_1h) - Q_b^n(\overline{e_1} \otimes e_1h)(\cdot - M^{-n}e_j) = \sum_{\alpha\in\mathbb{Z}^s} \nabla_{e_j} b_n(\alpha)(\overline{e_1} \otimes e_1h)(M^n \cdot -\alpha).$$

Note that the shifts of h are stable, therefore

$$\lim_{n\to\infty} \left\|\nabla_{e_j} b_n(\overline{e_1} \otimes e_1)\right\|_\infty = 0. \tag{3.1}$$

For $j = 2, \ldots, r$, we know that $e_1\chi$ and $(e_1 + e_j)\chi$ both satisfy the moment conditions of order 1, hence, $Q_a^n(e_1\chi)$ and $Q_a^n(e_1 + e_j)\chi$ converge to the same limit ϕ in the L_2-norm. This shows that, for $j = 2, \ldots, r$, $\left\|Q_a^n(e_j\chi)\right\|_2 \to 0$ as $n \to \infty$. By using (2.15), we have

$$vec((Q_a^n(e_j\chi)) \odot (Q_a^n(e_k\chi))^T) = Q_b^n((\overline{e_k} \otimes e_j)h), \quad j, k = 1, \ldots, r, \ \ n = 1, 2, \ldots .$$

By (2.3), we obtain

$$\lim_{n\to\infty} \left\|Q_b^n(\overline{e_k} \otimes e_j)h\right\|_\infty = 0, \quad (j, k) \neq (1, 1).$$

Since

$$Q_b^n(\overline{e_k}\otimes e_j)h=\sum_{\alpha\in\mathbb{Z}} b_n(\alpha)(\overline{e_k}\otimes e_j)h(M^n\cdot-\alpha),$$

it follows that

$$\lim_{n\to\infty}\left\|b_n(\overline{e_k}\otimes e_j)\right\|_\infty=0,\quad (j,k)\neq(1,1). \tag{3.2}$$

Since $\{e_j,\ j=1,\dots,r\}$ is a basis for C^r, it follows that $\{\overline{e_k}\otimes e_j,\ \ j,k=1,\dots,r\}$ is a basis for C^{r^2}. Then each $v\in V$ can be expressed as

$$v=\sum_{j=1}^{r}\sum_{k=1}^{r}\sum_{\beta\in\mathbb{Z}^s} d_{jk}(\beta)(\overline{e_k}\otimes e_j)\delta_\beta,$$

where $d_{jk}\in E_\mu$, $j,k=1,\dots,r$. Since $v\in V$, we have

$$0=(\overline{e_1^T}\otimes e_1^T)\sum_{\alpha\in\mathbb{Z}^s} v(\alpha)=(\overline{e_1^T}\otimes e_1^T)\sum_{j=1}^{r}\sum_{k=1}^{r}\sum_{\beta\in\mathbb{Z}^s} d_{jk}(\beta)(\overline{e_1^T}\otimes e_1^T)=\sum_{\beta\in\mathbb{Z}^s} d_{11}(\beta).$$

It follows from [12] that there exist $u_1,u_2,\dots,u_s\in\ell_1(\mathbb{Z}^s)$ such that $d_{11}=\sum_{j=1}^s\nabla_{e_j}u_j$. We have

$$T_b^n\left(\sum_{\beta\in\mathbb{Z}^s}\sum_{j=1}^{s}\nabla_{e_j}u_j(\beta)(\overline{e_1}\otimes e_1)\delta_\beta\right)(\alpha)$$
$$=\sum_{\gamma\in\mathbb{Z}^s} b_n(M^n\alpha-\gamma)\sum_{j=1}^{s}\nabla_{e_j}u_j(\gamma)(\overline{e_1}\otimes e_1),\quad \alpha\in\mathbb{Z}^s$$

and

$$T_b^n\left(\sum_{(j,k)\neq(1,1)}\sum_{\beta\in\mathbb{Z}^s} d_{jk}(\beta)(\overline{e_j}\otimes e_k)\delta_\beta\right)(\alpha)$$
$$=\sum_{(j,k)\neq(1,1)}\sum_{\gamma\in\mathbb{Z}^s} b_n(M^n\alpha-\gamma)(\overline{e_j}\otimes e_k)d_{jk}(\gamma),\quad \alpha\in\mathbb{Z}^s.$$

Therefore

$$\left\|T_b^n\left(\sum_{\beta\in\mathbb{Z}^s}\sum_{j=1}^{s}\nabla_{e_j}u_j(\beta)(\overline{e_1}\otimes e_1)\delta_\beta\right)\right\|_\infty\leqslant\sum_{j=1}^{s}\|\nabla_{e_j}b_n(\overline{e_1}\otimes e_1)\|_\infty\|u_j\|_1 \tag{3.3}$$

and

$$\left\|T_b^n\left(\sum_{(j,k)\neq(1,1)}\sum_{\beta\in\mathbb{Z}^s} d_{jk}(\beta)(\overline{e_j}\otimes e_k)\delta_\beta\right)\right\|_\infty\leqslant\|b_n(\overline{e_j}\otimes e_k)\|_\infty\sum_{(j,k)\neq(1,1)}\|d_{jk}\|_1. \tag{3.4}$$

Using (2.6), (3.1)–(3.4) and the expression of v, we prove that (1) of Theorem 3.1 holds.

To prove (2) of Theorem 3.1. We claim that V is invariant under T_b. Indeed, if not, then there exists $v \in V$ such that $T_b v$ is not in V. Note that the codimension of V in $E_\mu^{r^2}$ is 1. Hence, any $u \in E_\mu^{r^2}$ can be expressed as $u = w + c(T_b v)$ for some $w \in V$ and $c \in \mathbb{C}$. By (1) of Theorem 3.1, we have

$$\lim_{n\to\infty} \|T_b^n u\|_\infty = 0 \quad \forall u \in E_\mu^{r^2}.$$

Therefore, $\rho(T_b|_{E_\mu^{r^2}}) < 1$. On the other hand, by Theorem 2.4, we have $\rho(T_b|_{E_\mu^{r^2}}) \geqslant 1$. This contradiction shows that V is invariant under T_b. It follows from Theorem 2.3 that b satisfies the basic sum rule. Hence, a also satisfies the basic sum rule.

To establish the sufficiency part of the theorem. Let ϕ_0 be a vector of compactly supported functions in $(L_2(\mathbb{R}^s))^r$ such that ϕ_0 satisfies the moment conditions of order 1, and let $g_0 := Q_a\phi_0 - \phi_0$. To estimate $Q_a^{n+1}\phi_0 - Q_a^n\phi_0$, we observe that

$$Q_a^{n+1}\phi_0 - Q_a^n\phi_0 = Q_a^n(Q_a\phi_0 - \phi_0) = Q_a^n g_0. \tag{3.5}$$

Since $e_1^T \sum_{\alpha\in\mathbb{Z}^s} \phi_0(\cdot - \alpha) = 1$ and a satisfies the basic sum rule, we have

$$\begin{aligned} e_1^T \sum_{\alpha\in\mathbb{Z}^s} (Q_a\phi_0)(\cdot - \alpha) &= e_1^T \sum_{\alpha\in\mathbb{Z}^s}\sum_{\beta\in\mathbb{Z}^s} a(\beta)\phi_0(M\cdot - M\alpha - \beta) \\ &= \sum_{\beta\in\mathbb{Z}^s} e_1^T \left[\sum_{\alpha\in\mathbb{Z}^s} a(\beta - M\alpha)\right]\phi_0(M\cdot - \beta) = e_1^T \sum_{\beta\in\mathbb{Z}^s}\phi_0(M\cdot - \beta) = 1. \end{aligned}$$

Therefore, $Q_a\phi_0$ also satisfies the moment conditions of order 1. Consequently, for almost every $x \in \mathbb{R}^s$,

$$e_1^T \sum_{\alpha\in\mathbb{Z}^s} g_0(\cdot - \alpha) = 0.$$

By (2.2), we obtain

$$(\overline{e_1^T} \otimes e_1^T)\sum_{\alpha\in\mathbb{Z}^s} vec(g_0 \odot g_0^T)(\alpha) = (\overline{e_1^T} \otimes e_1^T)\sum_{\alpha\in\mathbb{Z}^s}\int_{\mathbb{R}^s} vec(g_0(\alpha + x)\overline{g_0(x)}^T)\,dx = 0.$$

Since $a \in E_\mu^{r\times r}$ and ϕ_0 is compactly supported, we have

$$\|vec(g_0 \odot g_0^T)\|_{E_\mu^{r^2}} < \infty.$$

Hence $vec(g_0 \odot g_0^T)$ lies in V. By (2.4), we have

$$\|Q_a^n g_0\|_2^2 \leqslant |vec((Q_a^n g_0) \odot (Q_a^n g_0)^T)(0)|.$$

Since

$$T_b^n(vec(g_0 \odot g_0^T))(\alpha) = \sum_{\beta\in\mathbb{Z}^s} b_n(M^n\alpha - \beta)vec(g_0 \odot g_0^T)(\beta),$$

then

$$\begin{aligned} T_b^n vec(g_0 \odot g_0^T)(0) &= \sum_{\beta\in\mathbb{Z}^s} b_n(-\beta)vec(g_0 \odot g_0^T)(\beta) = \sum_{\beta\in\mathbb{Z}^s} b_n(\beta)vec(g_0 \odot g_0^T)(-\beta) \\ &= Q_b^n vec(g_0 \odot g_0^T)(0) = vec((Q_a^n g_0) \odot (Q_a^n g_0)^T)(0). \end{aligned}$$

It follows that

$$\| Q_a^n g_0 \|_2^2 \leqslant \left| vec((Q_a^n g_0) \odot (Q_a^n g_0)^T)(0) \right|$$
$$= \left| T_b^n vec(g_0 \odot g_0^T)(0) \right| \leqslant \left\| T_b^n vec(g_0 \odot g_0^T) \right\|_\infty, \quad n = 1, 2, \ldots .$$

Since T_b is a compact operator, then there exists an eigenvalue τ of $T_b|_V$ such that $\rho(T_b|_V) = |\tau|$. We write $T_b v = \tau v$ for some $v \in V$ with $v \neq 0$. Therefore $T_b^n v = \tau^n v$, for $n = 1, 2, \ldots$. It follows from (1) of Theorem 3.1 that $\rho(T_b|_V) < 1$. Hence there exist positive constants C and $0 < \eta < 1$, such that

$$\| Q_a^n g_0 \|_2^2 \leqslant C\eta^n, \quad n = 1, 2, \ldots,$$

which implies that $Q_a^n \phi_0$ is a Cauchy sequence in $(L_2(\mathbb{R}^s))^r$. Let ψ_0 be another $r \times 1$ vector of $(L_{2,c}(\mathbb{R}^s))^r$ that satisfies the Strang–Fix conditions of order 1, then $e_1^T \sum_{\alpha \in \mathbb{Z}^s} (\phi_0 - \psi_0) = 0$. It follows from above discussions, $Q_a^n(\phi_0 - \psi_0)$ converges to 0 in the L_2-norm. Therefore, $Q_a^n \phi_0$ and $Q_a^n \psi_0$ converge to the same limit. Thus, the subdivision scheme associated with mask a and a general dilation matrix M converges in the L_2-norm. We complete the proof of Theorem 3.1. □

By the proof of Theorem 3.1, we have

Theorem 3.2. *Let $a \in E_\mu^{r \times r}$ for some $\mu > 0$, and $H(0) = \sum_{\alpha \in \mathbb{Z}^s} a(\alpha)/m$ satisfies* (1.3). *Let b be given by* (2.5) *and T_b be defined by* (2.6). *Then the subdivision scheme associated with mask a and a general dilation matrix M converges in the L_2-norm if and only if*

(1) *a satisfies the basic sum rule, and*
(2) $\rho(T_b|_V) < 1$,

where V is the linear space defined by (2.8).

When mask a is finitely supported, let W be the minimum invariant subspace of the transition operator T_b generated by $vec(e_1 e_1^T \Delta_j \delta)$, $j = 1, 2, \ldots, s$, $vec(e_2 e_2^T \delta), \ldots, vec(e_r e_r^T \delta)$, where Δ_j denotes the difference operator on $\ell_0(\mathbb{Z}^s)$ given by

$$\Delta_j u := 2u - u(\cdot - e_j) - u(\cdot + e_j), \quad u \in \ell_0(\mathbb{Z}^s).$$

It follows from Theorem 5.1 in [20] and Theorem 4.1 in [28] that

Theorem 3.3. *Suppose that a is finitely supported. Then the subdivision schemes associated with mask a and a general dilation matrix M converges in L_2-norm if and only if*

$$\rho(T_b|_W) < 1.$$

Remark 3.4. We remark that L_2-convergence of subdivision schemes associated with masks a having exponential decay was investigated in [12] with $r = 1$ and in [19] for the case $s = 1$ and $M = 2$. These theorems will also provide some sufficient conditions for the characterization of smoothness of multiple refinable functions associated with Eq. (1.1). Using a different approach, the L_2-convergence of subdivision schemes with an infinitely supported mask has also been considered in [14] with $r = 1$, $s = 1$ and $M = 2$ and in [13] for the case $s = 1$ and $M = 2$. When mask a is polynomially decaying, the L_2-solution of refinement equation with $s = 1$ and $M = 2$ was investigated in [30].

4. Biorthogonal multiple refinable functions

Linear independence and stability are two important concepts. The shifts of compactly supported functions $\varphi_1, \ldots, \varphi_r \in L_2(\mathbb{R}^s)$ are said to be linearly independent if

$$\sum_{j=1}^{r} \sum_{\alpha\in\mathbb{Z}^s} c_j(\alpha)\varphi_j(\cdot - \alpha) = 0$$

implies $c_j = 0$, $j = 1, 2, \ldots, r$. The shifts of $\varphi_1, \ldots, \varphi_r$ are linearly independent if and only if, for any $\xi \in \mathbb{C}^s$, the sequences $(\hat{\varphi}_j(\xi+2\beta\pi))_{\beta\in\mathbb{Z}^s}$, $j = 1, \ldots, r$ are linearly independent [22,30]. Hence linear independence implies stability.

Suppose $\varphi = (\varphi_1, \ldots, \varphi_r)^T$ and $\tilde{\varphi} = (\tilde{\varphi}_1, \ldots, \tilde{\varphi}_r)^T$ are dual vectors of compactly supported functions in $L_2(\mathbb{R}^s)$. It is easily seen that the shifts of φ and $\tilde{\varphi}$ are linearly independent. Thus, linear independence is a necessary condition for the existence of a dual vector of compactly supported functions.

Let $\varphi = (\varphi_1, \ldots, \varphi_r)^T$ be a L_2-solution of (1.1) with mask a being finitely supported such that the shifts of $\varphi_1, \ldots, \varphi_r$ are linearly independent, we want to find a dual $\tilde{\varphi}$ such that $\tilde{\varphi} = (\tilde{\varphi}_1, \ldots, \tilde{\varphi}_r)^T$ satisfies following refinement equation

$$\tilde{\varphi}(x) = \sum_{\alpha\in\mathbb{Z}^s} \tilde{a}(\alpha)\tilde{\varphi}(Mx - \alpha), \quad x \in \mathbb{R}^s, \tag{4.1}$$

where $\tilde{a}$ is a finitely supported sequence of $r \times r$ matrices on $\mathbb{Z}^s$.

When $s = 1$ and $M = 2$, Jia [19] proved that there exists a dual $\tilde{\varphi}$ of φ such that $\tilde{\varphi}$ satisfies (4.1) for some finite mask $\tilde{a}$. Thus, linear independence is a sufficient and necessary condition for the existence of a dual refinable vector of compactly supported functions for the case $s = 1$ and $M = 2$. In this section we shall show that under some mild assumptions on masks a and $\tilde{a}$, there exists a vector $\tilde{\varphi} = (\tilde{\varphi}_1, \ldots, \tilde{\varphi}_r)^T$ satisfying (4.1) such that $\tilde{\varphi}$ is dual to φ. For finite mask $\tilde{a}$, let

$$\tilde{H}(\xi) := \frac{1}{|\det M|} \sum_{\alpha\in\mathbb{Z}^s} \tilde{a}(\alpha)e^{-i\alpha\cdot\xi}, \quad \xi \in \mathbb{R}^s.$$

We assume that $\tilde{H}(0)$ also satisfies eigenvalue condition. Let Ω be a complete set of representatives of the distinct cosets of the quotient group $\mathbb{Z}^s/M^T\mathbb{Z}^s$ with $0 \in \Omega$. It was proved in [31] that if φ is dual to $\tilde{\varphi}$, then

$$\sum_{\omega\in\Omega} H(\xi + (M^T)^{-1}2\pi\omega)\tilde{H}(\xi + (M^T)^{-1}2\pi\omega)^* = I_r, \tag{4.2}$$

for all $\xi \in \mathbb{R}^s$, where $\tilde{H}(\xi + (M^T)^{-1}2\pi\omega)^*$ denotes the complex conjugate transpose of $\tilde{H}(\xi + (M^T)^{-1}2\pi\omega)$ and I_r is the $r \times r$ identity matrix.

Therefore, to find a vector refinable function $\tilde{\varphi}$ satisfying (4.1) such that $\tilde{\varphi}$ is dual to φ, one must solve (4.2). However, there is no general method to solve this equation. By using block centrally symmetric matrices, Chen, Micchelli and Xu proved that for a large family of masks a and $\tilde{a}$, Eq. (4.2) is solvable (see Theorem 3.1 of [2]). In this section, we always assume that (4.2) is solvable.

Theorem 4.1. *Let a be finitely supported and* $\varphi = (\varphi_1, \ldots, \varphi_r)^T$ *be a* L_2*-solution of* (1.1) *with linearly independent shifts. Suppose that* (4.2) *is solvable. Then there exists a refinable vector* $\tilde{\varphi} = (\tilde{\varphi}_1, \ldots, \tilde{\varphi}_r)$ *of compactly supported functions in* $L_2(\mathbb{R}^s)$ *such that* $\tilde{\varphi}$ *is dual to* φ.

Proof. The proof of Theorem 4.1 follows the line [19]. Let

$$G(\xi) := ([\varphi_j, \varphi_k](e^{-i\xi}))_{1 \leqslant j,k \leqslant r},$$

where the bracket product of φ_j and φ_k is defined by

$$[\varphi_j, \varphi_k](e^{-i\xi}) := \sum_{\beta \in \mathbb{Z}^s} \hat{\varphi}_j(\xi + 2\pi\beta)\overline{\hat{\varphi}_k(\xi + 2\pi\beta)}, \quad \xi \in \mathbb{R}^s.$$

Since the shifts of $\varphi_1, \ldots, \varphi_r$ are linearly independent, it follows from [22,25] that the matrix $G(\xi)$ is positive definite for every $\xi \in \mathbb{R}^s$. Denote $\tilde{\varphi} = (\tilde{\varphi}_1, \ldots, \tilde{\varphi}_r)^T$ by

$$\hat{\tilde{\varphi}}(\xi) := G(\xi)^{-1}\hat{\varphi}(\xi), \quad \xi \in \mathbb{R}^s.$$

It is easy to check that the shifts of $\tilde{\varphi}_1, \ldots, \tilde{\varphi}_r$ are stable. For every $\xi \in \mathbb{R}^s$, we have

$$\begin{aligned}\sum_{\beta \in \mathbb{Z}^s} \hat{\tilde{\varphi}}(\xi + 2\pi\beta)\hat{\varphi}(\xi + 2\pi\beta)^* &= G(\xi)^{-1} \sum_{\beta \in \mathbb{Z}^s} \hat{\varphi}(\xi + 2\pi\beta)\hat{\varphi}(\xi + 2\pi\beta)^* \\ &= G(\xi)^{-1}G(\xi) = I_r,\end{aligned}$$

which implies that $\tilde{\varphi}$ is dual to φ. Note that

$$\begin{aligned}\hat{\tilde{\varphi}}(\xi) &= G(\xi)^{-1}H((M^T)^{-1}\xi)\hat{\varphi}((M^T)^{-1}\xi) \\ &= G(\xi)^{-1}H((M^T)^{-1}\xi)G((M^T)^{-1}\xi)\hat{\tilde{\varphi}}((M^T)^{-1}\xi).\end{aligned}$$

Let $\tilde{H}(\xi) = G(M^T\xi)^{-1}H(\xi)G(\xi)$, $\xi \in \mathbb{R}^s$, then

$$\hat{\tilde{\varphi}}(\xi) = \tilde{H}((M^T)^{-1}\xi)\hat{\tilde{\varphi}}((M^T)^{-1}\xi).$$

Hence, $\tilde{\varphi}$ is a vector refinable functions. Clearly, $\tilde{H}(\xi)$ is 2π-periodic. We write $\tilde{H}(\xi) = (\tilde{h}_{jk}(\xi))_{1 \leqslant j,k \leqslant r}$, where

$$\tilde{h}_{jk}(\xi) = \sum_{\alpha \in \mathbb{Z}^s} \tilde{a}_{jk}(\alpha)e^{-i\alpha\cdot\xi}/m, \quad \xi \in \mathbb{R}^s.$$

It follows from the definition of $\tilde{H}(\xi)$ that there exists some $\mu > 0$ such that $\tilde{a} = (\tilde{a}_{jk})_{1 \leqslant j,k \leqslant r} \in E_\mu^{r\times r}$ for all $j, k = 1, \ldots, r$. Since $\tilde{\varphi}$ is dual to φ, we have that the subdivision scheme associated with $\tilde{a}$ and dilation matrix M is convergent in the L_2-norm. Let $\tilde{b} \in E_\mu^{r^2\times r^2}$ be denoted as follows:

$$\tilde{b}(\alpha) := \sum_{\beta \in \mathbb{Z}^s} \overline{\tilde{a}(\beta)} \otimes \tilde{a}(\alpha + \beta)/m, \quad \alpha \in \mathbb{Z}^s$$

and let $\tilde{V}$ be the linear space given by (2.8). It follows from Theorem 3.2 that $\tilde{a}$ satisfies the basic sum rule and $\rho(T_{\tilde{b}}|_{\tilde{V}}) < 1$. For $L = 1, 2, \ldots,$ we can find $\tilde{a}_L \in (\ell_0(\mathbb{Z}^s))^{r\times r}$ such that $\|\tilde{a}_L - \tilde{a}\|_{E_\mu^{r\times r}} \to 0$ as $L \to \infty$. Let

$$\tilde{H}_L(\xi) := \sum_{\alpha \in \mathbb{Z}^s} \tilde{a}_L(\alpha)e^{-i\alpha\cdot\xi}/m, \quad \xi \in \mathbb{R}^s$$

and

$$e_L(\xi) := I_r - \sum_{\omega\in\Omega} H(\xi + (M^T)^{-1}2\pi\omega)\tilde{H}_L(\xi + (M^T)^{-1}2\pi\omega)^* \tag{4.3}$$

for all $\xi \in \mathbb{R}^s$. Since (4.2) is solvable for mask a, there exists an $r \times r$ matrix of trigonometric polynomial $F(\xi)$ such that $H(\xi)$ and $F(\xi)$ satisfy (4.2). Denote

$$F_L(\xi) = \tilde{H}_L(\xi) + e_L(\xi)^* F(\xi), \quad \xi \in \mathbb{R}^s.$$

Then $F_L(\xi)$ is an $r \times r$ matrix of trigonometric polynomial. Write

$$F_L(\xi) = \sum_{\alpha\in\mathbb{Z}^s} f_L(\alpha)e^{-i\alpha\cdot\xi}/m, \quad \xi \in \mathbb{R}^s,$$

where $f_L \in (\ell_0(\mathbb{Z}^s))^{r\times r}$. Since $\|\tilde{a}_L - \tilde{a}\|_{E_\mu^{r\times r}} \to 0$ as $L \to \infty$, by the construction of F_L, we have that $\|f_L - \tilde{a}\|_{E_\mu^{r\times r}} \to 0$ as $L \to \infty$. Therefore, we may choose $\tilde{a}_L$ $(L = 1, 2, \ldots)$ in such a way that each f_L satisfies the basic sum rule. For sufficiently large L, 1 is a simple eigenvalue of $\sum_{\alpha\in\mathbb{Z}^s} f_L(\alpha)/m$ and its other eigenvalues are less than 1 in modulus.

With the help of the following identity (see [20])

$$\sum_{\omega\in\Omega} e^{-i\alpha\cdot 2\pi(M^T)^{-1}} = \begin{cases} m & \text{if } \alpha = M\beta \text{ for some } \beta \in \mathbb{Z}^s, \\ 0 & \text{if } \alpha \notin M\mathbb{Z}^s. \end{cases}$$

We have $e_L(\xi + 2\pi(M^T)^{-1}\omega) = e_L(\xi)$ for any $\omega \in \Omega$. It follows from (4.2) and (4.3) that

$$\begin{aligned} &\sum_{\omega\in\Omega} H(\xi + 2\pi(M^T)^{-1}\omega)F_L(\xi + 2\pi(M^T)^{-1}\omega)^* \\ &\quad = \sum_{\omega\in\Omega} H(\xi + 2\pi(M^T)^{-1}\omega)\tilde{H}_L(\xi + 2\pi(M^T)^{-1}\omega)^* \\ &\qquad + \sum_{\omega\in\Omega} H(\xi + 2\pi(M^T)^{-1}\omega)F(\xi + 2\pi(M^T)^{-1}\omega)^* e_L(\xi) = I_r. \end{aligned}$$

Let $\tilde{b}_L$ be denoted by

$$\tilde{b}_L(\alpha) := \sum_{\beta\in\mathbb{Z}^s} \overline{f_L(\beta)} \otimes f_L(\alpha + \beta)/m, \quad \alpha \in \mathbb{Z}^s. \tag{4.4}$$

It follows from above discussion that $\tilde{b}_L \to \tilde{b}$ in the space $E_\mu^{r^2\times r^2}$ as $L \to \infty$. Note that $\rho(T_{\tilde{b}}|_{\tilde{V}}) < 1$, where $\tilde{V}$ is the linear space defined by (2.8), therefore $\rho(T_{\tilde{b}_L}|_{\tilde{V}}) < 1$, for sufficiently large L. By Theorem 3.2, subdivision scheme associated with f_L and dilation matrix M converges in the L_2-norm. It follows from Theorem 3.1 in [31] that limit function φ_L is an $r \times 1$ refinable vector of compactly supported functions in $(L_2(\mathbb{R}^s))^r$. Furthermore, φ_L is dual to φ. □

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