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Harnack estimates for curvature flows depending on mean curvature

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Abstract. We prove the Harnack estimates of curvature flows of hypersurfaces in $\mathbf{R}^{n+1}$, where the normal velocity is given by a smooth function f depending only on the mean curvature. By use of the estimates, we get some corollaries including the integral Harnack inequality. In particular, we give the conditions, with which the solution to the flow is a translation soliton and an expanding soliton respectively.

§1 Introduction

It is well-known that there are many results about Harnack estimates for some geometric flows. We also call it LYH inequality, which is first noted by Peter Li and S.-T.Yau for the scalar heat flow in [9] . Hamilton studied the Ricci flow on surface[5] and got such estimate. He also got the matrix Harnack inequality in the scalar heat flow[6], and such estimates in Ricci flow for all dimensions[7] and mean curvature flow[8]. Ben Chow obtained similar inequalities for Gauss curvature flow[2] and Yamabe flow[3]. Moreover, Ben Andrews treated a class of geometric flows by the inverse map of Guass map in [1]. Wang Jie got the estimate for H^k-flow in [11].

Let M^n be a smooth manifold without boundary, and let F_0: $M^n \to \mathbf{R}^{n+1}$ be a smooth immersion which is convex. We consider a smooth evolving one-parameter family of hypersurface immersions described by a map $F(\cdot, t) : M^n \times [0, T) \to \mathbf{R}^{n+1}$, where the evolution is given by the following equation:

$$\begin{cases} \frac{\partial}{\partial t} F(x,t) = f(H(x,t))\overrightarrow{\nu} \\ F(x,0) = F_0(x), \quad \forall x \in M^n, \end{cases}$$

where $\overrightarrow{\nu}$ is the unit inward normal and f is a smooth function depending only on the mean curvature H.

If $f = H$ we get the mean curvature flow, and for $f = H^k$ it is the H^k-flow. In this paper, we call it f-flow. Ben Andrews also considered a class of geometric flows and his results hold on the compact case. K.Smoczyk proved the short time existence of smooth admissible solution

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to the f-flow under the condition $f' > 0$ and the Harnack inequality on compact hypersurfaces of $\mathbf{R}^{n+1}$ in [10]. The purpose of this paper is to prove the same estimate for the complete case.

Throughout this paper, we denote M_t as the admissible solution to the flow, and $t \in [0, T]$ on which the solution exists.

We always assume the solution satisfies the following condition

$(\star)$ compact or complete with bounded $|A|, |DA|, |D^2A|$, at each time t,

where A is the second fundamental form of M_t.

Theorem 1.1. Assume that F_0: $M^n \to \mathbf{R}^{n+1}$ is an admissible smooth and convex immersion, M_t is convex under the condition $(\star)$ and that $f : [0, +\infty) \to \mathbf{R}$ is a smooth function such that for all $x \in [0, +\infty)$ we have

$$f' > 0, \frac{f''}{f'}x^2 \geq ax, (\frac{f''}{f'}x)' \leq 0, ff''x + ff' - (f')^2 x \geq 0$$

where $a \in \mathbf{R}$ is a constant. Then we can find a small positive constant d such that

$$\frac{\partial}{\partial t}f + 2Df(V) + A(V, V) + cf'H \geq 0$$

holds for all tangent vectors V as long as $t \in [0, T]$ and $d + (a + 2)t > 0$, where we have set $c(t) := \frac{1}{d+(a+2)t}$.

It is just the differential Harnack inequality for the f-flow. As usual we integrate it over paths in space-time to get an integral Harnack inequality.

Corollary 1.1. Under the assumption of Theorem 1.1, for $\forall\ 0 \leq t_1 < t_2 < T$ satisfying $d + (a + 2)t_i > 0, (i = 1, 2)$ and $Y_1, Y_2 \in M$, we have

$$H(Y_2, t_2) \geq \left(\frac{d + (a + 2)t_1}{d + (a + 2)t_2}\right)^{\frac{1}{a+2}} e^{-\frac{\triangle}{4D}} H(Y_1, t_1)$$

where

$$D = \inf_{\substack{x \in M \\ t_1 \leq t \leq t_2}} f'(H(x, t)), \quad \textit{and} \quad \triangle = \inf \int \left|\frac{dY}{dt}\right|_M^2 dt$$

is the infimum over all paths $Y(t)$ remaining on the surface at time t with $Y = Y_1$ at $t = t_1$ and $Y = Y_2$ at $t = t_2$, $\frac{dY}{dt}$ is the velocity vector of the path, and $\left|\frac{dY}{dt}\right|_M$ is the length of its component tangent to the surface M. In particular, if $f(0) > 0$, then $\triangle \leq \frac{d(Y_1, \widehat{Y}_2, t_1)^2}{t_2 - t_1}$, where $d(Y_1, \widehat{Y}_2, t_1)$is the distance along the surface at time t_1 between Y_1 and $\widehat{Y}_2$, such that $\widehat{Y}_2$ evolves normally to Y_2 at t_2.

We say a solution is eternal if it is defined for $-\infty < t < +\infty$. Eternal solutions arise as limits of dilations (in space-time) of slowly forming singularities. One interesting class of eternal solutions is the translation solitons. These are surfaces which evolve by translating in space with a constant velocity $T = V + f\overrightarrow{\nu}$. So the Harnack inequality is also held on translation solitons. For any strictly convex eternal solutions to mean curvature flow and H^k-flow, we have known they must be translation solitons when the mean curvature attains its maximum value at a point in space-time. For the f-flow, we have the similar result in the following.

Theorem 1.2. If f satisfies the assumptions of Theorem 1.1 and $a + 2 > 0$, then any strictly convex eternal solution under the condition $(\star)$ to the f-flow where the mean curvature attains its maximum value at a point in space-time, it must be a translation soliton.

Translation solitons are steady solitons, which exist for $-\infty < t < +\infty$. There are also homothetically expanding solitons and shrinking solitons. In fact, the homothetic solution is closely related to the Harnack inequality, because it becomes an equality in this case. In [10], for a homothetic solution we have $D_a\widetilde{V}_b = fh_{ab} + cg_{ab}$. For $c > 0$, the solution is expanding soliton, and for $c < 0$, it is shrinking. B.L.Chen proved the type III singularity of mean curvature flow must be the expanding gradient soliton in [4]. Similarly we have the following result for f-flow.

Theorem 1.3. If f satisfies the assumptions of Theorem 1.1 and $a + 2 > 0$, then any strictly convex solution under the condition $(\star)$ to the f-flow which exists for $0 < t < +\infty$, where $(d + (a+2)t)H^{a+2}$ attains its maximum value at a point in space-time, it must be an expanding soliton.

§2 Notations and evolution equations

Suppose M is an n-dimensional manifold without boundary immersed in Euclidean space $\mathbf{R}^{n+1}$, it is parametrized locally by $X = \{x^i\}$ in $\mathbf{R}^n$, where $i = 1, ..., n$. On M a point $Y = \{y^\alpha\}_{\alpha=1,\dots,n+1}$ in $\mathbf{R}^{n+1}$ is given locally by $y^\alpha = F^\alpha(x^i)$. Then the tangent vectors on M in $\mathbf{R}^{n+1}$ is denoted by $D_iY = \frac{\partial Y}{\partial x^i}$. The Euclidean metric is $I = \{I_{\alpha\beta}\}$, then the induced metric $G = \{g_{ij}\}$ on M is

$$g_{ij} = I(D_iY, D_jY) = I_{\alpha\beta}D_iy^\alpha D_jy^\beta.$$

The unit normal $\overrightarrow{\nu} = \{N^\alpha\}$ is defined by

$$I_{\alpha\beta}N^\alpha N^\beta = 1 \qquad and \qquad I_{\alpha\beta}N^\alpha D_iy^\beta = 0.$$

On the convex surfaces we choose $\overrightarrow{\nu}$ to be inward. The metric $G = \{g_{ij}\}$ induces a Levi-Civita connection $\Gamma = \{\Gamma^i_{jk}\}$ on M. So we can take covariant derivatives $D = \{D_i\}$ of tensors on M. The covariant derivative of 1-form D_jy^α is

$$D_iD_jy^\alpha = h_{ij}N^\alpha$$

where $A = \{h_{ij}\}$ be the second fundamental form of M. Its trace $H = g^{ij}h_{ij}$ is the mean curvature.

By assumption we have

$$\frac{\partial}{\partial t}F(\cdot, t) = f\overrightarrow{\nu},$$

As [10] we formally derive the evolution equations for various geometric objects on M, these are:

$$\begin{aligned} \frac{\partial g_{ij}}{\partial t} &= -2fh_{ij}, \\ \frac{\partial h_{ij}}{\partial t} &= f'\Delta h_{ij} + f''D_iHD_jH - (f + f'H)h_{ik}h^k_j + f'|A|^2h_{ij}. \end{aligned}$$

We have to calculate many evolution equations. To avoid too complicated formulas it is most convenient to work with coordinates associated to a moving frame. We use similar moving orthonormal frame coordinates as [10].

Let $\{E_a\}, (a = 1, ..., n)$ be an orthonormal frame locally, where $E_a = E^i_a\frac{\partial}{\partial x^i}$ is tangent to M. To keep the vectors orthonormal and tangent to M_t under the flow, we let

$$\frac{\partial}{\partial t}E^i_a = fg^{ij}h_{jl}E^l_a.$$

We can write the components of tensors in terms of the frame, for example $h_{ab} = A(E_a, E_b)$ or in local coordinates, $h_{ab} = h_{ij}E_a^i E_b^j$. We always denote indices $a, b, c\dots$ under orthonormal frame, and $i, j, k\dots$ under local coordinates.

Now we take covariant derivatives under moving orthonormal frame, for example $D_aV_b = E_a^i E_b^j D_iV_j$. Then the Laplacian is $\Delta = \sum_a D_aD_a$. We define the operator $\Box \doteqdot f'\Delta$.

In addition, we also define the time like vector field D_t on the frame bundle as [10], which differentiates in the direction of the moving frame. In local coordinates

$$\begin{aligned} D_tV_a &= \{\frac{\partial}{\partial t}V_k + fg^{ij}h_{jk}V_i\}E_a^k \\ &= \frac{\partial}{\partial t}V_a + fh_a^iV_i = \frac{\partial}{\partial t}V_a + fh_{ac}V_c. \end{aligned}$$

By direct computations, we have

$$\begin{aligned} D_tg_{ab} &= 0; \\ D_th_{ab} &= D_t(E_a^iE_b^jh_{ij}) = f'\Delta h_{ab} + f''D_aHD_bH \\ &\quad +(f - f'H)h_{ac}h_{cb} + f'|A|^2h_{ab}; \\ D_tf &= f'(\Delta f + f|A|^2). \end{aligned}$$

We can also calculate some formulas of the commutator of derivations, which are useful for the computation of Section 3.

Formula 2.1. If $V = \{V_a\}$ is a covector on M, then

$$D_tD_aV_b - D_aD_tV_b = fh_{ac}D_cV_b + (h_{ac}D_bf - h_{ab}D_cf)V_c.$$

Formula 2.2. If g is a smooth function on M, then we have commutator relations,

$$\begin{aligned} (1).(D_t - \Box)D_ag - D_a(D_t - \Box)g &= f''D_aH\Delta g + f'h_{ad}h_{dc}D_cg \\ &\quad +(f - f'H)h_{ac}D_cg; \\ (2).(D_t - \Box)\Box g - \Box(D_t - \Box)g &= f''\Delta f\Delta g + ff''|A|^2\Delta g \\ &\quad +2ff'h_{ac}D_aD_cg + 2f'h_{ac}D_afD_cg \\ &\quad +(f - f'H)D_afD_ag. \end{aligned}$$

Formula 2.3.

$$D_aD_bf = D_th_{ab} - fh_{ac}h_{bc}.$$

The proofs of these formulas are similar in [7,8,11]. We leave the details as an exercise.

§3 The computation

In this section we can get the evolution equation of the basic Harnack expression under the f-flow. To this end we define the following basic quantities, which all vanish on a translation soliton.

Definition 3.1. We let

$$\begin{aligned} &X_a = D_af + h_{ab}V_b, \qquad Y_{ab} = D_aV_b - fh_{ab}, \\ &Z = D_tf + 2V_aD_af + h_{ab}V_aV_b, \quad W_{ab} = D_th_{ab} + V_cD_ch_{ab}, \\ &W = D_tf + V_cD_cf, \qquad P = D_tH + V_cD_cH, \end{aligned}$$

$$U_a = (D_t - \Box)V_a + f'h_{ab}D_bf.$$

Then now we can take the computation.

Theorem 3.1. For any solution to f-flow and any vector field V, we have:

$$\begin{aligned}(D_t - \Box)Z &= f'|A|^2Z + 2X_aU_a - 2f'h_{bc}Y_{ab}Y_{ac} - 4f'W_{ab}Y_{ab} + f''P^2 \\ &\quad +(f - f'H)|X|^2.\end{aligned}$$

Proof. The computation is very tedious but direct,

$$\begin{aligned}(D_t - f'\Delta)Z &= (D_t - f'\Delta)(D_tf + 2V_aD_af + h_{ab}V_aV_b) \\ &= \{f'D_tf|A|^2 + 2f'V_aD_af|A|^2 + f'|A|^2h_{ab}V_aV_b\} \\ &\quad +\{2f'h_{ac}D_afD_cf + 2D_af(D_t - f'\Delta)V_a \\ &\quad +2f'h_{ad}h_{dc}V_aD_cf + 2h_{ab}V_a(D_t - f'\Delta)V_b\} \\ &\quad +\{-2f^2f'h_{ac}h_{ab}h_{bc} + 4ff'D_bV_ah_{ac}h_{bc} - 2f'h_{ab}D_cV_aD_cV_b\} \\ &\quad +\{4ff'h_{ab}D_th_{ab} + 4ff'V_ah_{cd}D_ah_{cd} - 4f'D_bV_aD_th_{ab} \\ &\quad -4f'D_ch_{ab}V_aD_cV_b\} \\ &\quad +\{f''D_tHD_tH + 2f''V_aD_aHD_tH + f''D_aHD_bHV_aV_b\} \\ &\quad +\{(f - f'H)D_cfD_cf + 2V_a(f - f'H)h_{ac}D_cf + (f - f'H)h_{ac}h_{bc}V_aV_b\} \\ &= f'|A|^2Z + 2X_aU_a - 2f'h_{bc}Y_{ab}Y_{ac} - 4f'W_{ab}Y_{ab} + f''P^2 \\ &\quad +(f - f'H)|X|^2. \qquad \sharp\end{aligned}$$

Now we make a little change of these quantities by adding the factors of $\frac{1}{t}$. We can refer to K. Smocyzk's paper [10] and make the following definition.

Definition 3.2. Again we let

$$\widetilde{X_a} = X_a, \quad \widetilde{Y_{ab}} = Y_{ab} - cg_{ab}, \quad \widetilde{Z} = Z + cf'H, \quad \widetilde{W_{ab}} = W_{ab} + ch_{ab},$$
$$\widetilde{W} = W + cf'H, \quad \widetilde{P} = P + cH, \quad \widetilde{U_a} = U_a - \frac{\partial}{\partial t}(\ln c)V_a,$$

where c is given in Theorem 1.1.

Here we denote that $f'\widetilde{P} = \widetilde{W} = \widetilde{Z} - \widetilde{X_a}V_a$. By direct computation, we can get the evolution equation of $cf'H$.

$$(D_t - \Box)(cf'H) = \frac{\partial}{\partial t}(\ln c)cf'H + cf(f''H + f')|A|^2 - c(\frac{f''}{f'}H)'|Df|^2.$$

Combining it with Theorem 3.1, we obtain:

Corollary 3.1. For any solution to f-flow and for any vector field V we have

$$\begin{aligned}(D_t - \Box)\widetilde{Z} &= (f'|A|^2 - 2(c\frac{f''}{f'}H + 2c))\widetilde{Z} + 2\widetilde{X_a}\widetilde{U_a} - 2f'h_{bd}\widetilde{Y_{ab}}\widetilde{Y_{ad}} - 4f'\widetilde{W_{ab}}\widetilde{Y_{ab}} + f''\widetilde{P}^2 \\ &\quad +(f - f'H)|\widetilde{X}|^2 + 2(c\frac{f''}{f'}H + \frac{\partial}{\partial t}(\ln c) + 2c)\widetilde{X_a}V_a - c(\frac{f''}{f'}H)'|Df|^2 \\ &\quad +cf'H(c\frac{f''}{f'}H + \frac{\partial}{\partial t}(\ln c) + 2c) + c(ff''H + ff' - (f')^2H)|A|^2.\end{aligned}$$

The evolution of $\widetilde{Z}$ is identical with the result in [10].

§4 Proof of Theorem 1.1

Under the condition $(\star)$, we have $|A| \leq B_0(T_0), |DA| \leq B_1(T_0), |D^2A| \leq B_2(T_0)$ on closed interval $[0, T_0]$, where $T_0 < T$. Here $B_0(T_0), B_1(T_0), B_2(T_0)$ are positive constants depending on T_0, and $|\cdot|$ is the norm for tensors with respect to $\{g_{ij}(t)\}$. We give the following lemma first.

Lemma 4.1. If $f : [0, +\infty) \to \mathbf{R}$ is a smooth function, there exists a function $\varphi(x, t) > 0$, s.t. $\varphi(x, t) \to \infty$ as $x \to \infty$, and $(D_t - f'(H)\Delta_{g(t)})\varphi \geq C\varphi$ on $M \times [0, T_0]$ for any $C > 0$.

Proof. As [11], we set $\varphi(x, t) = \varepsilon e^{Bt} g_0(x)$ for $\varepsilon > 0$ and B will be chosen later, where $g_0(x) \in C^\infty(M)$ such that $g_0(x) \geq 1$ everywhere and $g_0(x) \to \infty$, as $x \to \infty$, and $|\Delta_{g(t)} g_0(x)| \leq C_0$ for a positive constant C_0 depending on n, B_0, B_1, T_0, for $t \in [0, T_0]$.

We get $|f'\Delta_{g(t)} g_0(x)| \leq C_1 g_0(x)$, since f is smooth function and H is bounded on $M \times [0, T_0]$. Therefore

$$(D_t - f'\Delta_{g(t)})\varphi \geq (B - C_1)\varphi \geq C\varphi,$$

if $B \geq C_1 + C$. $\sharp$

Now we prove Theorem 1.1.

Proof. For the compact case, K.Smoczyk had proven the estimate in [10]. For the complete case under the condition $(\star)$, we will show the inequality holding on any closed interval $[0, T_0]$, where $T_0 \leq T$ and on which $d + (a + 2)t > 0$ holds.

Let $\varphi(x, t) = \varepsilon_1 e^{Bt} g_0(x)$ be the function defined in Lemma 4.1, and $\psi(t) = \varepsilon_2 e^{Lt}$ a function depending only on t, where $\varepsilon_1, \varepsilon_2 > 0$ are small constants, and B, L will be chosen later. Set $\widehat{Z} = \widetilde{Z} + \varphi + \psi|V|^2$,

$$D_t f = (f')^2 \Delta H + f'f''|DH|^2 + ff'|A|^2 \geq -C,$$
$$|D_a f| = |fD_a H| \leq C,$$

where we use the facts that f is a smooth function and $|H|$, $|\Delta H|$ and $|DH|$ are bounded on $M \times [0, T_0]$. And

$$h_{ab} + \psi g_{ab} \geq \varepsilon_2 g_{ab} > 0,$$

so $\widetilde{Z} + \psi|V|^2$ has lower bound on $M \times [0, T_0]$ for any V.

From those above, we can choose d sufficiently small such that $\widehat{Z} > \delta > 0$ for $t = 0$ and for all tangent vectors V or where it is out of a compact set $\Omega \subset M$. We assume if $\widehat{Z}$ first attains zero at $(x_0, t_0) \in \Omega \times [0, T_0]$, with the direction V. Then

$$0 = \frac{\partial \widehat{Z}(V + sW)}{\partial s}|_{s=0}(x_0, t_0) = 2W_a(\widetilde{X_a} + \psi V_a),$$

for any W, so we get $\widetilde{X_a} = -\psi V_a$. We extend V to a vector field in space-time such that $\widetilde{Y_{ab}} = 0, \widetilde{U}_a = 0$, then at the point we have

$$\begin{aligned}
0 \geq\ & (D_t - \Box)\widehat{Z} \geq (f'|A|^2 - 2(c\frac{f''}{f'}H + 2c))\widetilde{Z} + 2\widetilde{X_a}\widetilde{U_a} - 2f'h_{bc}\widetilde{Y_{ab}}\widetilde{Y_{ac}} - 4f'\widetilde{W_{ab}}\widetilde{Y_{ab}} \\
& + \frac{f''}{(f')^2}(\widetilde{Z} - \widetilde{X_a}V_a)^2 + (f - f'H)|\widetilde{X}|^2 + 2(c\frac{f''}{f'}H + \frac{\partial}{\partial t}(\ln c) + 2c)\widetilde{X_a}V_a + (D_t - \Box)\varphi \\
& + (D_t\psi)|V|^2 \\
\geq\ & (f'|A|^2 - 2(c\frac{f''}{f'}H + 2c))(-\varphi - \psi|V|^2) + \frac{f''}{(f')^2}\varphi^2 - 2(c\frac{f''}{f'}H + \frac{\partial}{\partial t}(\ln c) + 2c)\psi|V|^2
\end{aligned}$$

$$
\begin{aligned}
&+(f-f'H)\psi^2|V|^2+C\varphi+L\psi|V|^2\\
=\ &(C-f'|A|^2+2(c\frac{f''}{f'}H+2c)+\frac{f''}{(f')^2}\varphi)\varphi\\
&+(L-f'|A|^2-2\frac{\partial}{\partial t}(\ln c)+(f-f'H)\psi)\psi|V|^2>0.
\end{aligned}
$$

The second inequality is because f satisfies the conditions of the theorem and $c>0$. The last inequality is because f smoothly depends on H and $|A|$ is bounded on $M\times[0,T_0]$, and we choose ε_1 and ε_2 sufficiently small and B and L large enough. So it is the contradiction. We get $\widehat{Z}>0$ on $M\times[0,T_0]$. We take $\varepsilon_1,\varepsilon_2\to 0$, then $\widetilde{Z}\geq 0$. ♯

Example 1. For the H^k-flow we have $f(H)=H^k, a\leq k-1$, and hence we choose $a=k-1$, the Harnack estimate becomes

$$\frac{\partial H^k}{\partial t}+2DH^k(V)+A(V,V)+\frac{kH^k}{(k+1)t+d}\geq 0.$$

It has been obtained by Wang Jie[11]. When $k=1$, it has been obtained by Hamilton[8].

Example 2. When $f(H)=\sqrt{1+H^2}$ and we choose $a=0$, the function f satisfies the conditions of Theorem 1.1. The Harnack estimate of the f-flow is

$$\frac{\partial H}{\partial t}+2DH(V)+\frac{f}{H}A(V,V)+\frac{H}{2t+d}\geq 0.$$

§5 Some corollaries

We first give the proof of Corollary 1.1.

Proof. Along any path $Y(t)=F(X(t),t)$, we have

$$\frac{df}{dt}=f'(\frac{\partial H}{\partial t}+DH(\frac{dX}{dt})),$$

and hence from the Harnack estimate we obtain, by taking $V=\frac{1}{2}\frac{dX}{dt}$

$$\frac{df}{dt}\geq-\frac{1}{4}A(\frac{dX}{dt},\frac{dX}{dt})-cf'H.$$

Because $f'>0$ and $A(\frac{dX}{dt},\frac{dX}{dt})\leq H|\frac{dX}{dt}|^2$ for a convex surface, we arrive at the inequality

$$\frac{d}{dt}\ln H\geq-\frac{1}{4f'}\left|\frac{dX}{dt}\right|^2-c.$$

Note that $\frac{dX}{dt}$ is the tangential component of $\frac{dY}{dt}$, so that

$$\ln\frac{H(Y_2,t_2)}{H(Y_1,t_1)}\geq-\frac{1}{a+2}\ln\frac{d+(a+2)t_2}{d+(a+2)t_1}-\frac{\triangle}{4D}$$

where

$$D=\inf_{\substack{x\in M\\ t_1\leq t\leq t_2}}f'(H(x,t)), and\quad \triangle=\inf\int\left|\frac{dY}{dt}\right|_M^2dt.$$

The result follows by exponentiating.

In particular, if $f(0)\geq 0$, then $f\geq 0$ for all convex solution since $f'>0$. Moreover, $\frac{\partial g_{ij}}{\partial t}=-2fh_{ij}$ and $h_{ij}\geq 0$, the metric g_{ij} will be weakly shrinking. Hence, $\left|\frac{dY}{dt}\right|_M\leq\frac{d(Y_1,\widetilde{Y_2},t_1)}{t_2-t_1}$. ♯

Corollary 5.1(Nondecreasing of tH)**.** Under the assumption of Theorem 1.1, if $a+2\geq 1$, then for any two times $0<t_1\leq t_2<T$ and $\forall\ x\in M$ at t_2, we have:

$$H(x,t_2)\geq\frac{t_1}{t_2}H(\widetilde{x},t_1)$$

which $\widetilde{x}$ at t_1 evolves normally to x at t_2.

Proof. Take $V = 0$. We have

$$0 \le D_t f + cf'H = f'(\frac{\partial H}{\partial t} + cH) = \frac{f'}{t}(t\frac{\partial H}{\partial t} + \frac{t}{d+(a+2)t}H) \le \frac{f'}{t}(t\frac{\partial H}{\partial t} + H).$$

So $\frac{\partial(tH)}{\partial t} \ge 0$. ♯

Corollary 5.2. Under the assumption of Theorem 1.1, if a strictly convex solution to f-flow exists in $(-\infty, 0]$ under the condition $(\star)$ and $a + 2 > 0$, then

$$\frac{\partial H}{\partial t} \ge 0.$$

Proof. For any $\alpha > 0$, the solution exists on $[-\alpha, 0]$, and $H > 0$, then we have

$$f'(\frac{\partial H}{\partial t} + \frac{H}{(a+2)(t+\alpha)}) \ge f'(\frac{\partial H}{\partial t} + \frac{H}{d+(a+2)(t+\alpha)}) \ge 0.$$

We take limit as $\alpha \to +\infty$. ♯

§6 Translation soliton

In this section and next we suppose that f satisfies the assumption of Theorem 1.1 and $a + 2 > 0$. If the solution under the condition $(\star)$ is strictly convex and eternal, then it exists on any closed interval $[-C, C], C > 0$, so

$$\widetilde{Z} = Z + \frac{f'H}{d+(a+2)(t+C)} \ge 0.$$

As $C \to \infty$, we have $Z \ge 0$, for all V.

Lemma 6.1. Under the condition $(\star)$, if $F \ge 0$ is a weakly positive function on M, such that $(D_t - \Box)F = 0$, and if $Z \ge F$ at $t = 0$ for all V, then $Z \ge F$ at all subsequent times for all V.

Proof. For the compact case, we set $\widetilde{Z} = Z - F + \varepsilon e^{Bt}$; for the complete case, set $\widetilde{Z} = Z - F + \varphi + \psi|V|^2$, where φ, ψ are the functions mentioned in Section 4. Now we may prove $\widetilde{Z} \ge 0$ as in the proof of Theorem 1.1. ♯

Lemma 6.2. If $(D_t - \Box)F = 0$, and $F(x_0, 0) > 0$, $x_0 \in M$, then $F > 0$, for $t > 0$ and everywhere on M.

Proof. Because $H > 0$ everywhere and $f' > 0$, so $\Box$ is a strictly elliptic operator. Therefore the strong maximum principle holds. ♯

From the above lemmas, we get, if there is a point where $Z > 0$ for all V at $t = t_0$, then $Z > 0$ every where on M, for $t > t_0$ and for all V.

We assume H attains its maximum at (x_0, t_0), then at this point

$$\frac{\partial f}{\partial t} = 0, D_a f = 0.$$

So $Z(x_0, t_0) = \frac{\partial f}{\partial t} + 2V_a D_a f + h_{ab} V_a V_b = 0$ when $V = 0$. The strong maximum principle (Lemma 6.2) implies that prior to that time Z has a zero vector V at each point in space-time. Since $Z \ge 0$ and $h_{ab} > 0$, the zero vector V is unique and varies smoothly. By the first variation of Z, we can obtain the zero vector $V_a = -h_{ab}^{-1} D_b f$.

Additionally, We can extend V to be a vector field in space-time satisfying

$$Y_{ab} = -W_{ad} h_{bd}^{-1}.$$

Note $X_a = 0$ at that point, and $0 = Z = V_a X_a + f'P$, so we also have $P = 0$ at that point. Then we have

$$0 \geq (D_t - \Box)Z = 2f' h_{bd}^{-1} W_{ab} W_{ad} \geq 0,$$

then $W_{ab} = 0$. So we get a vector field V on M_t such that on each point $X_a = 0, W_{ab} = 0$ with V.

We will get Theorem 1.2 from the following theorem.

Theorem 6.1. If on a strictly convex solution to f-flow, a vector field satisfies

$$D_a f + h_{ab} V_b = 0, \tag{6.1}$$

and

$$D_t h_{ab} + V_c D_c h_{ab} = 0, \tag{6.2}$$

then the solution must be a translation soliton.

Proof. Differentiate (6.1), it is

$$D_a D_b f + V_c D_a h_{bc} + h_{bc} D_a V_c = 0. \tag{6.3}$$

By the Formula 2.3

$$D_a D_b f = D_t h_{ab} - f h_{ac} h_{bc}. \tag{6.4}$$

From (6.2),(6.3) and (6.4), we have $f h_{ac} h_{bc} = h_{bc} D_a V_c$. Since $h_{ab} > 0$, it means $\{h_{ab}\}$ is invertible, we get

$$f h_{ac} = D_a V_c. \tag{6.5}$$

We consider the vector field $T = V + f\overrightarrow{\nu}$, where $\overrightarrow{\nu}$ is the unit inward normal. Differentiating it, we have

$$D_i T = (D_i V^j - f h_i^j) \frac{\partial X}{\partial x^j} + (V^j h_{ij} + D_i f)\overrightarrow{\nu} = 0,$$

the last equality is because (6.1) and (6.5). So T is a constant vector and the solution is a translation soliton. ♯

§7 Expanding soliton

If the solution under the condition ($\star$) is strictly convex and exists for $0 < t < +\infty$, then by Theorem 1.1 we have $\widetilde{Z} \geq 0$, for all V.

We assume $(d + (a+2)t)H^{a+2}$ attains its maximum at (x_0, t_0), then at this point $\widetilde{Z} = 0$ when $V = 0$. By Corollary 3.1, the strong maximum principle also holds in this case. It implies there must be a zero vector V such that $\widetilde{Z} = 0$ at each point (x, t), for $t < t_0$. Since $\widetilde{Z} \geq 0$ and $h_{ab} > 0$, we can obtain the zero vector $V_a = -h_{ab}^{-1} D_b f$ by using the first variation of $\widetilde{Z}$.

Similarly, We extend V to be a vector field in space-time satisfying

$$\widetilde{Y_{ab}} = -\widetilde{W_{ad}} h_{bd}^{-1}.$$

Note $\widetilde{X_a} = 0$ at that point, and $0 = \widetilde{Z} = V_a \widetilde{X_a} + f'\widetilde{P}$, so we also have $\widetilde{P} = 0$ at that point. Then we have

$$0 \geq (D_t - \Box)\widetilde{Z} \geq 2f' h_{bd}^{-1} \widetilde{W_{ab}} \widetilde{W_{ad}} \geq 0,$$

then $\widetilde{W_{ab}} = 0$. So we get a vector field V on M_t such that on each point $\widetilde{X_a} = 0, \widetilde{W_{ab}} = 0$ with V.

We will get Theorem 1.3 from the following theorem.

Theorem 7.1. If on a strictly convex solution to f-flow, a vector field satisfies

$$D_a f + h_{ab} V_b = 0, \tag{7.1}$$

and

$$D_t h_{ab} + V_d D_d h_{ab} + c h_{ab} = 0, \tag{7.2}$$

where $c > 0$, then the solution must be an expanding soliton.

Proof. Differentiate (7.1), it is

$$D_a D_b f + V_d D_a h_{bd} + h_{bd} D_a V_d = 0. \tag{7.3}$$

By the Formula 2.3

$$D_a D_b f = D_t h_{ab} - f h_{ad} h_{bd}. \tag{7.4}$$

From (7.2),(7.3) and (7.4), we have

$$f h_{ad} h_{bd} + c h_{ab} = h_{bd} D_a V_d.$$

Since $h_{ab} > 0$, we obtain $f h_{ad} + c g_{ad} = D_a V_d$. So the solution is an expanding soliton. ♯

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