Time-periodic universes

De-Xing Kong[1], Kefeng Liu[2,3] and Ming Shen[2]
[1]*Department of Mathematics, Zhejiang University, Hangzhou 310027, China*
[2]*Center of Mathematical Sciences, Zhejiang University, Hangzhou 310027, China*
[3]*Department of Mathematics, University of California at Los Angeles, CA 90095, USA*

In this letter we construct a new time-periodic solution of the vacuum Einstein's field equations whose Riemann curvature norm takes the infinity at some points. We show that this solution is intrinsically time-periodic and describes a time-periodic universe with the "black hole". New physical phenomena are investigated and new singularities are analyzed for this universal model.

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1. Introduction. The Einstein's field equations are the fundamental equations in general relativity and play an essential role in cosmology. The exact solutions of the Einstein's field equations play a crucial role in the study of general relativity and cosmology. Typical examples are the Schwarzschild solution and Kerr solution (see [13] and [7]). Although many interesting and important solutions have been obtained (see, e.g., [1] and [14]), there are still several fundamental open problems. One such problem is *if there exists a "time-periodic" solution, which contains physical singularities such as black hole, to the Einstein's field equations.* This letter aims to solves this problem.

For the evolutionary equations, time-periodic or stationary solutions correspond to the late time behavior of solutions for a large class of initial data. In the general theory of relativity, the time-periodic "black hole" solutions (if they exist) seem to provide reasonable candidates for the final state of gravitational collapse. As pointed out that in [3], such solutions can be defined as those invariant with respect to an isometry of the domain of outer communications which takes every point to its future, or more generally, such that points sufficiently close to infinity are mapped to their future. The study of the periodic solutions to the Einstein's field equations was initiated in Papapetrou [11]-[12]. See also the important paper [6]. Dafermos [3] proved a theorem about the non-existence of spherically symmetric black-hole space-times with time-periodicity outside the event horizon, other than Schwarzschild in the vacuum case and Reissner-Nordström in the case of electromagnetic fields and matter sources of a particular kind. This important result generalizes the "no-hair" theorem from the static to the time-periodic case. Up to now, very few results on the well-posedness for the Einstein's field equations have been established. In their classical monograph [2], Christodoulou and Klainerman proved the global nonlinear stability of the Minkowski space for the vacuum Einstein's field equations, i.e., they showed the global nonlinear stability of the trivial solution of the vacuum Einstein's field equations. Lindblad and Rodnianski [10] proved the global stability of the Minkowski space for the vacuum Einstein's field equations in wave coordinate gauge for the set of restricted data coinciding with the Schwarzschild solution in the neighborhood of space-like infinity. This work provides a new and simple approach to the stability problem originally solved by Christodoulou and Klainerman. In the Ph.D. thesis [16], Zipser generalized the result of Christodoulou and Klainerman [2] to the Einstein-Maxwell equations. In a series of interesting papers (see [4]-[5]), Finster, Kamran, Smoller and Yau investigated the non-existence of time-periodic solutions of the Dirac equation, the Einstein-Dirac-Maxwell equations or the Einstein-Dirac-Yang/Mills equations.

The first exact time-periodic solution of the vacuum Einstein's field equations was constructed by the authors in [8]. The solution presented in [8] is time-periodic, and describes a regular space-time, which has vanishing Riemann curvature tensor but is inhomogenous, anisotropic and not asymptotically flat. In our recent work [9], we construct several kinds of new time-periodic solutions of the vacuum Einstein's field equations whose Riemann curvature tensors vanish, keep finite or take the infinity at some points in these space-times, respectively. However the norm of Riemann curvature tensors of all these solutions vanishes. This implies that these solutions essentially describe regular time-periodic space-times.

In this letter, we construct a new time-periodic solution of the vacuum Einstein's field equations. For this solution, not only its Riemann curvature tensor takes the infinity at some points, but also the norm of the Riemann curvature tensor also go to the infinity at these points. Therefore, this solution possesses some physical singularities. We also show that this solution is intrinsically time-periodic and then can be used to describe a time-periodic universe with the "black hole". New physical phenomena are investigated and new singularities are analyzed for this universal model. This gives a positive answer of the above open problem.

2. Time-periodic solution. Consider the following vacuum Einstein's field equations

$$G_{\mu\nu} \stackrel{\triangle}{=} R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = 0, \tag{1}$$

or equivalently,

$$R_{\mu\nu} = 0, \tag{2}$$

where $g_{\mu\nu}$ $(\mu,\nu = 0,1,2,3)$ is the unknown Lorentzian metric, $R_{\mu\nu}$ is the Ricci curvature tensor, R is the scalar curvature and $G_{\mu\nu}$ is the Einstein tensor.

Take (t,r,θ,φ) as the spherical coordinates with $t \in \mathbb{R}$, $r \in [0,\infty)$, $\theta \in [0,2\pi)$, $\varphi \in [-\pi/2,\pi/2]$ and let $x^0 = t$, $x^1 = r$, $x^2 = \theta$, $x^3 = \varphi$. In the coordinates (t,r,θ,φ), we consider the metric of the form

$$ds^2 = g_{\mu\nu} = u^2dt^2 + 2qdtdr + 2vdtd\varphi - a^2b^2dr^2 - a^2d\theta^2, \tag{3}$$

where u, v, a are smooth functions of t, r, and b, q are smooth functions of t. It is easy to verify that the determinant of $(g_{\mu\nu})$ is given by

$$g \stackrel{\triangle}{=} \det(g_{\mu\nu}) = -a^4b^2v^2. \tag{4}$$

For the metric (3), a direct calculation gives

$$R_{02},\ R_{12},\ R_{13},\ R_{23},\ R_{33} = 0. \tag{5}$$

On the other hand,

$$R_{03} = -\frac{v_{rr}}{2a^2b^2}. \tag{6}$$

Noting (2), we have

$$R_{03} = 0, \quad \text{i.e.,} \quad \frac{v_{rr}}{2a^2b^2} = 0. \tag{7}$$

Solving (7) leads to

$$v = cr + d, \tag{8}$$

where $c = c(t)$ and $d = d(t)$ are integral functions depending on t. For simplicity, let $d = 0$. Then (8) becomes

$$v = cr. \tag{9}$$

Substituting (9) into (3) and computing R_{11}, R_{22} yields

$$R_{11} = -\frac{a^2 + 2raa_r - 2r^2aa_{rr} + 2r^2a_r^2}{2r^2a^2} \tag{10}$$

and

$$R_{22} = \frac{aa_r + raa_{rr} - ra_r^2}{ra^2b^2}. \tag{11}$$

Noting (2), we have $R_{22} = 0$. Solving it gives

$$a = fr^g, \tag{12}$$

where f, g are two integral functions depending on t. Noting (2) again yields $R_{11} = 0$ and substituting (12) into the equation $R_{11} = 0$ leads to

$$g = -\frac{1}{4}. \tag{13}$$

Thus, (12) becomes

$$a = fr^{-\frac{1}{4}}. \tag{14}$$

Substituting (9) and (14) into (3) and computing R_{01}, we have

$$R_{01} = -\frac{4bf_t + fb_t}{rfb}. \tag{15}$$

Noting (2), we have $R_{01} = 0$, i.e.,

$$\frac{4bf_t + fb_t}{rfb} = 0.$$

Solving this equation gives

$$b = \frac{n}{f^4}, \tag{16}$$

where n is integral constant. Without loss of generality, we may assume $n = 1$. Then (16) becomes

$$b = \frac{1}{f^4}. \tag{17}$$

We now calculate the term R_{00}.

Substituting (9), (14) and (17) into (3), by a direct calculation we obtain

$$R_{00} = -\frac{A}{2r^3cf^2}, \tag{18}$$

where

$$\begin{aligned} A \;=\;& r^{\frac{3}{2}}f^8cu^2 - 2r^{\frac{5}{2}}f^8cuu_r + 2r^{\frac{7}{2}}f^8cu_r^2 + \\ & 2r^{\frac{7}{2}}f^8cuu_{rr} + 4r^3ff_{tt}c - 4r^3ff_tc_t - 24r^3f_t^2c. \end{aligned} \tag{19}$$

Noting (2) again, we have $R_{00}=0$. Solving this equation gives

$$u^2 = 4Hr^{\frac{3}{2}} + H_0 r\ln r + H_1 r, \tag{20}$$

where H_0 and H_1 are two integral functions depending on t, and H is given by

$$H = \frac{24cf_t^2 + 4c_t ff_t - 4cff_{tt}}{f^8 c}. \tag{21}$$

Summarizing the above discussion, we can obtain the following theorem.

Theorem 1 *The vacuum Einstein's filed equations (1) have the following solutions in the coordinates* (t,r,θ,φ)

$$ds^2 = (dt,dr,d\theta,d\varphi)(g_{\mu\nu})(dt,dr,d\theta,d\varphi)^T, \tag{22}$$

where

$$(g_{\mu\nu}) = \begin{pmatrix} 4Hr^{\frac{3}{2}} + H_0 r\ln r + H_1 r & q & 0 & cr \\ q & -\frac{1}{f^6\sqrt{r}} & 0 & 0 \\ 0 & 0 & -\frac{f^2}{\sqrt{r}} & 0 \\ cr & 0 & 0 & 0 \end{pmatrix}, \tag{23}$$

in which H_0*,* H_1*,* c*,* q *and* f *are arbitrary functions of* t*, and* H *is defined by (21).* ■

In particular, taking

$$H_0 = H_1 = 0, \quad c = b = \frac{1}{f^4} \tag{24}$$

and

$$f = 1+\sin t, \quad q = 0, \tag{25}$$

we have

Theorem 2 *The vacuum Einstein's filed equations (1) have the following time-periodic solution in the coordinates* (t,r,θ,φ)

$$ds^2 = (dt,dr,d\theta,d\varphi)(\eta_{\mu\nu})(dt,dr,d\theta,d\varphi)^T, \tag{26}$$

where

$$\begin{cases} \eta_{00} = \dfrac{16r^{\frac{3}{2}}(1+\sin t+\cos^2 t)}{(1+\sin t)^8}, \\ \eta_{03} = \dfrac{r}{(1+\sin t)^4}, \\ \eta_{11} = -\dfrac{1}{\sqrt{r}(1+\sin t)^6}, \\ \eta_{22} = -\dfrac{(1+\sin t)^2}{\sqrt{r}}, \\ \eta_{01}=\eta_{02}=\eta_{12}=\eta_{13}=\eta_{23}=\eta_{33}=0. \end{cases} \quad ■ \tag{27}$$

Proof. By Theorem 1, it is obvious that the metric (26) is a solution of the vacuum Einstein's filed equations (1). It suffices to prove the solution (26) is time-periodic. To do so, we prove that the variable t is a time coordinate.

It is easy to verify that the determinant of $(\eta_{\mu\nu})$ is given by

$$\eta \triangleq \det(\eta_{\mu\nu}) = -\frac{r}{(1+\sin t)^{12}}. \tag{28}$$

Obviously, $t = 2k\pi - \pi/2$ $(k\in\mathbb{Z})$ and $r=0$ are the singularities of the space-time described by (26). A detailed analysis on these singularities will be given in next section.

When $t \neq 2k\pi - \pi/2$ $(k\in\mathbb{Z})$ and $r\neq 0$, it holds that

$$\eta_{00} = \frac{16r^{\frac{3}{2}}(1+\sin t+\cos^2 t)}{(1+\sin t)^8} > 0,$$

$$\begin{vmatrix} \eta_{00} & \eta_{01} \\ \eta_{01} & \eta_{11} \end{vmatrix} = -\frac{16r(1+\sin t+\cos^2 t)}{(1+\sin t)^{14}} < 0,$$

$$\begin{vmatrix} \eta_{00} & \eta_{01} & \eta_{02} \\ \eta_{01} & \eta_{11} & \eta_{12} \\ \eta_{20} & \eta_{21} & \eta_{22} \end{vmatrix} = \frac{16\sqrt{r}(1+\sin t+\cos^2 t)}{(1+\sin t)^{12}} > 0$$

and

$$\begin{vmatrix} \eta_{00} & \eta_{01} & \eta_{02} & \eta_{03} \\ \eta_{10} & \eta_{11} & \eta_{12} & \eta_{13} \\ \eta_{20} & \eta_{21} & \eta_{22} & \eta_{23} \\ \eta_{30} & \eta_{31} & \eta_{32} & \eta_{33} \end{vmatrix} = -\frac{r}{(1+\sin t)^{12}} < 0.$$

This implies that the variable t is a time coordinate. Therefore, (26) is indeed a time-periodic solution of the vacuum Einstein's field equations (1). Thus, the proof of Theorem 2 is completed. □

3. Singularities. This section is devoted to the analysis of singularities of the time-periodic solution (26) of the vacuum Einstein's field equations.

By direct calculations, the Riemann curvature tensor of (26) reads

$$R_{2121} = \frac{(1+\sin t)^2}{4r^{\frac{5}{2}}}, \tag{29}$$

$$R_{0101} = \frac{2(1+\sin t)\sin t - 2\cos^2 t}{\sqrt{r}(1+\sin t)^8}, \tag{30}$$

$$R_{0221} = \frac{3(1+\sin t)\cos t}{2r^{\frac{3}{2}}}, \tag{31}$$

$$R_{0301} = \frac{3\cos t}{2(1+\sin t)^5}, \tag{32}$$

$$R_{0303} = -\frac{\sqrt{r}}{4(1+\sin t)^2}, \tag{33}$$

$$R_{0232} = \frac{(1+\sin t)^4}{8r}, \tag{34}$$

$$R_{0202} = \frac{2(1+\sin t)\sin t + 10\cos^2 t}{\sqrt{r}}, \tag{35}$$

$$R_{0131} = \frac{1}{8r(1+\sin t)^4}, \tag{36}$$

and the other $R_{\alpha\beta\mu\nu} = 0$. Moreover,

$$\mathbf{R} \triangleq R^{\alpha\beta\gamma\delta}R_{\alpha\beta\gamma\delta} = \frac{3(1+\sin t)^{12}}{4r^3}. \tag{37}$$

Therefore, when $t \neq 2k\pi - \pi/2$ ($k \in \mathbb{Z}$) and $r \to 0+$, it holds that

$$\mathbf{R} \longrightarrow +\infty. \tag{38}$$

Thus, we have

Proposition 1 *$r = 0$ is an essential singular point. Thus, the solution (26) describes a time-periodic space-time with a "black hole".* ■

In particular, when $t = 2k\pi - \pi/2$ ($k \in \mathbb{Z}$), we have $\mathbf{R} = 0$. This implies that the "black hole" disappears at these points.

According to the definition of the event horizon (see [15]), the hypersurfaces $t = 2k\pi - \pi/2$ ($k \in \mathbb{Z}$) are the event horizons of the space-time described by (26). Therefore we have

Proposition 2 *The solution (26) also contains non-essential singularities which consist of the hypersurfaces $t = 2k\pi - \pi/2$ ($k \in \mathbb{Z}$). These hypersurfaces correspond to the event horizons.*

Remark 1 *Notice that the curvature tensors are intrinsic and independent of the choice of the coordinates. It follows from (29)-(37) that the Lorentzian metric (26) is NOT the Minkowski metric written in some periodic coordinate system. On the other hand, it is well known that, by taking a static or stationary vacuum metric and performing a nontrivial periodic coordinate transformation one could produce many apparently periodic solutions - but which are not intrinsically periodic; however the metric (26) is NOT this case because of (29)-(37). In other words, our solution (26) is intrinsically time-periodic.*

We now investigate the behavior of the null curves and light-cones in the space-time (26).

Fixing θ and φ, we get the induced metric

$$ds^2 = \eta_{00}dt^2 + \eta_{11}dr^2.$$

Consider the null curves in the (t, r)-plan, which are defined by

$$\eta_{00}dt^2 + \eta_{11}dr^2 = 0.$$

Noting (27) gives

$$\frac{dt}{dr} = \pm\frac{\sqrt{1+\sin t}}{4r\sqrt{2-\sin t}}.$$

Thus, the null curves and light-cones are shown in Figure 1.

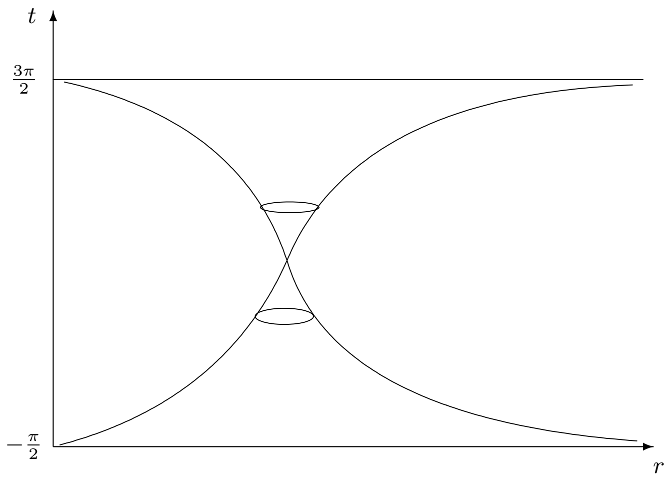

FIG. 1: Null curves and light-cones in the domains $-\pi/2 < t < 3\pi/2$.

We next study the geometric behavior of the t-slices.

For any fixed $t \in \mathbb{R}$, it follows from (26) that the induced metric of the t-slice reads

$$ds^2 = -\frac{1}{\sqrt{r}(1+\sin t)^2}[dr^2 + (1+\sin t)^8 d\theta^2]. \tag{39}$$

As mentioned before, the hypersurfaces $t = 2k\pi - \pi/2$ ($k \in \mathbb{Z}$) are singularities of the space-time described by (26), while, when $t \neq 2k\pi - \pi/2$ ($k \in \mathbb{Z}$), the t-slice is a three-dimensional cone-like manifold centered at $r = \infty$.

4. Summary and discussion. In our previous works [8]-[9], we have constructed three kinds of new time-periodic solutions of the vacuum Einstein's field equations: the regular time-periodic solution with vanishing Riemann curvature tensor, the regular time-periodic solution with finite Riemann curvature tensor and the time-periodic solution with physical singularities. However, the norm of the Riemann curvature tensors of all these solutions vanishes, therefore these solutions essentially describe some regular time-periodic space-times, these space-times contain some non-physical singularities, but no physical singularity.

In this letter we construct a new time-periodic solution (26) of the vacuum Einstein's field equations. The norm of the Riemann curvature tensor of the metric (26) goes to the infinity when r tends to zero. Therefore $r = 0$ is a physical singularity of the space-time described by (26), which is named as "black hole" in this letter. This solution also contains some non-essential singularities which consist of the hypersurfaces $t = 2k\pi - \pi/2 \quad (k \in \mathbb{Z})$. In particular, by (29)-(37) we observe that the metric (26) is impossible to be the Minkowski metric written in some periodic coordinate system; on the other hand, by taking a static or stationary vacuum metric and performing a nontrivial periodic coordinate transformation one could NOT produce the metric (26), this is to say, the solution (26) is intrinsically time-periodic. As a corollary, we would like to point out that the space-time (26) has a time-periodic time-like Killing vector field. Moreover, new physical phenomena have been investigated for the time-periodic universal model characterized by (26). Consequently, the solution (26) solves the long-time open problem mentioned at the first paragraph in Section 1, and more applications of this new space-time in modern cosmology and general relativity can be expected.

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