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Time-Periodic Solutions of the Einstein's Field Equations II: Geometric Singularities

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Abstract In this paper, we construct several kinds of new time-periodic solutions of the vacuum Einstein's field equations whose Riemann curvature tensors vanish, keep finite or take the infinity at some points in these space-times, respectively. The singularities of these new time-periodic solutions are investigated and some new physical phenomena are discovered.

Keywords: Einstein's field equations, time-periodic solution, Riemann curvature tensor, singularity, event horizon

MSC(2000): 83C05, 83C15, 83C75, 35C05

1 Introduction

This work is a continuation of our previous work [3] "Time-Periodic Solutions of the Einstein's Field Equations I: General Framework". As in [3], we still consider the time-periodic solutions of the following vacuum Einstein's field equations

$$G_{\mu\nu} \triangleq R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 0, \tag{1}$$

or equivalently,

$$R_{\mu\nu} = 0, \tag{2}$$

where $g_{\mu\nu}$ $(\mu, \nu = 0, 1, 2, 3)$ is the unknown Lorentzian metric, $R_{\mu\nu}$ is the Ricci curvature tensor, R is the scalar curvature and $G_{\mu\nu}$ is the Einstein tensor.

It is well known that the exact solutions of the Einstein's field equations play a crucial role in general relativity and cosmology. Typical examples are the Schwarzschild solution and Kerr solution. Although many interesting and important solutions have been obtained (see, e.g., [1] and [5]), there are still many fundamental open problems. One such problem is *if there exists a "time-periodic" solution, which contains physical singularities such as black hole, to the vacuum Einstein's field equations*. This paper continues the discussion of this problem.

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The first time-periodic solution of the vacuum Einstein's field equations was constructed by the first two authors in [3]. The solution presented in [3] is time-periodic, and describes a regular space-time, which has vanishing Riemann curvature tensor but is inhomogenous, anisotropic and not asymptotically flat. In particular, this space-time does not contain any essential singularity, but contains some non-essential singularities which correspond to steady event horizons, time-periodic event horizon and has some interesting new physical phenomena.

In this paper, we focus on finding the time-periodic solutions, which contain geometric singularities (see Definition 1 below) to the vacuum Einstein's field equations (1). We shall construct three kinds of new time-periodic solutions of the vacuum Einstein's field equations (1) whose Riemann curvature tensors vanish, keep finite or go to the infinity at some points in these space-times respectively. The singularities of these new time-periodic solutions are investigated and new physical phenomena are found. Moreover, the applications of these solutions in modern cosmology and general relativity may be expected. In the forthcoming paper [4], we shall construct a time-periodic solution of the Einstein's field equations with physical singularities (see also Definition 1 below), which describes a time-periodic universe with many new and interesting physical phenomena.

The paper is organized as follows. In §2 we present our procedure of finding new solutions of the vacuum Einstein's field equations and introduce the concepts of "geometric singularity" and "physical singularity". In §3 we construct three kinds of new time-periodic solutions of the vacuum Einstein's field equations whose Riemann curvature tensors vanish, keep finite or take the infinity at some points in these space-times, respectively. In this section, the singularities of these new time-periodic solutions are also investigated and some new physical phenomena are found and discussed. A summary and some discussions are given in §4.

2 Procedure of finding new solutions

We consider the metric of the following form

$$(g_{\mu\nu}) = \begin{pmatrix} u & v & p & 0 \\ v & 0 & 0 & 0 \\ p & 0 & f & 0 \\ 0 & 0 & 0 & h \end{pmatrix}, \tag{3}$$

where u, v, p, f and h are smooth functions of the coordinates (t, x, y, z). It is easy to verify that the determinant of $(g_{\mu\nu})$ is given by

$$g \overset{\triangle}{=} \det(g_{\mu\nu}) = -v^2 fh. \tag{4}$$

Throughout this paper, we assume that

$$g < 0. \tag{H}$$

Without loss of generality, we may suppose that f and g keep the same sign, for example,

$$f < 0 \ \ (resp.\ f > 0) \quad \text{and} \quad h < 0 \ \ (resp.\ g > 0). \tag{5}$$

In what follows, we solve the Einstein's field equations (2) under the framework of the Lorentzian metric of the form (3).

By a direct calculation, we have the Ricci tensor

$$R_{11} = -\frac{1}{2}\left\{\frac{v_x}{v}\left(\frac{f_x}{f}+\frac{h_x}{h}\right)+\frac{1}{2}\left[\left(\frac{f_x}{f}\right)^2+\left(\frac{h_x}{h}\right)^2\right]-\left(\frac{f_{xx}}{f}+\frac{h_{xx}}{h}\right)\right\}. \tag{6}$$

It follows from (2) that

$$\frac{v_x}{v}\left(\frac{f_x}{f}+\frac{h_x}{h}\right)+\frac{1}{2}\left[\left(\frac{f_x}{f}\right)^2+\left(\frac{h_x}{h}\right)^2\right]-\left(\frac{f_{xx}}{f}+\frac{h_{xx}}{h}\right)=0. \tag{7}$$

This is an ordinary differential equation of first order on the unknown function v. Solving (7) gives

$$v = V(t,y,z)\exp\left\{\int \Theta(t,x,y,z)dx\right\}, \tag{8}$$

where

$$\Theta = \left[\frac{f_{xx}}{f}+\frac{h_{xx}}{h}-\frac{1}{2}\left(\frac{f_x}{f}\right)^2-\frac{1}{2}\left(\frac{h_x}{h}\right)^2\right]\frac{fh}{(fh)_x},$$

and $V = V(t,y,z)$ is an integral function depending on t, y and z. Here we assume that

$$(fh)_x \neq 0. \tag{9}$$

In particular, taking the ansatz

$$f = -K(t,x)^2, \quad h = N(t,y,z)K(t,x)^2 \tag{10}$$

and substituting it into (8) yields

$$v = VK_x. \tag{11}$$

By the assumptions (H) and (9), we have

$$V \neq 0, \quad K \neq 0, \quad K_x \neq 0. \tag{12}$$

Noting (10) and (11), by a direct calculation we obtain

$$R_{13} = -\frac{V_z K_x}{KV}. \tag{13}$$

It follows from (2) that

$$R_{13} = 0.$$

Combining (12) and (13) gives

$$V_z = 0. \tag{14}$$

This implies that the function V depends only on t, y but is independent of x and z. Noting (10)-(11) and using (14), we calculate

$$R_{12} = -\frac{1}{2V}\left(\frac{p_{xx}}{K_x}-\frac{K_{xx}p_x}{K_x^2}-\frac{2pK_x}{K^2}+\frac{2K_xV_y}{K}\right). \tag{15}$$

Solving p from the equation $R_{12}=0$ yields

$$p = AK^2 + V_y K + \frac{B}{K}, \tag{16}$$

where A and B are integral functions depending on t, y and z. Noting (10)-(11) and using (14) and (16), we observe that the equation $R_{23}=0$ is equivalent to

$$B_z - 2K^3 A_z = 0. \tag{17}$$

Since K is a function depending only on $t,\ x$, and $A,\ B$ are functions depending on $t,\ y$ and z, we can obtain that

$$B = 2K^3 A + C(t,x,y), \tag{18}$$

where C is an integral function depending on $t,\ x$ and y. For simplicity, we take

$$A = B = C = 0. \tag{19}$$

Thus, (16) simplifies to

$$p = V_y K. \tag{20}$$

From now on, we assume that the function N only depends on y, that is to say,

$$N = N(y). \tag{21}$$

Substituting (10)-(11), (14) and (20)-(21) into the equation $R_{02}=0$ yields

$$u_x V_y + V(u_{yx} - 4V_y K_{xt}) = 0. \tag{22}$$

Solving u from the equation (22) leads to

$$u = 2K_t V. \tag{23}$$

Noting (10)-(11), (14), (20)-(21) and (23), by a direct calculation we obtain

$$R_{03} = 0, \tag{24}$$

$$\left\{\begin{array}{lcl} R_{22} & = & (4N^2V^2)^{-1}\left[2NV^2N_{yy} - 4N^2VV_{yy} + 4N^2V_y^2 - 2NVN_yV_y - V^2N_y^2\right], \\ R_{33} & = & -(4NV^2)^{-1}\left[2NV^2N_{yy} - 4N^2VV_{yy} + 4N^2V_y^2 - 2NVN_yV_y - V^2N_y^2\right] \end{array}\right. \tag{25}$$

and

$$R_{00} = (2KNV^2)^{-1}\left[4NV_tV_y^2 + 2NV^2V_{tyy} - 2NVV_tV_{yy} - 4NVV_yV_{ty} - VN_yV_tV_y + V^2N_yV_{ty}\right]. \tag{26}$$

Therefore, under the assumptions mentioned above, the Einstein's field equations (2) are reduced to

$$-\frac{N_{yy}}{N} + \frac{1}{2}\left(\frac{N_y}{N}\right)^2 + 2\frac{V_{yy}}{V} + \frac{N_yV_y}{NV} - 2\left(\frac{V_y}{V}\right)^2 = 0 \tag{27}$$

and

$$4V_y^2V_t + 2V^2V_{yyt} - 2VV_{yy}V_t - 4VV_yV_{yt} - \frac{VV_yV_tN_y}{N} + \frac{V^2V_{yt}N_y}{N} = 0. \tag{28}$$

On the other hand, (27) can be rewritten as

$$2\left(\frac{V_y}{V}\right)_y + \frac{V_y N_y}{VN} - \left(\frac{N_y}{N}\right)_y - \frac{1}{2}\left(\frac{N_y}{N}\right)^2 = 0 \tag{29}$$

and (28) is equivalent to

$$2\left(\frac{V_y}{V}\right)_{yt} + \left(\frac{V_y}{V}\right)_t \frac{N_y}{N} = 0. \tag{30}$$

Noting (21) and differentiating (29) with respect to t gives (30) directly. This shows that (29) implies (30). Hence in the present situation, the Einstein's field equations (2) are essentially (29). Solving V from the equation (29) yields

$$V = w(t)|N(y)|^{1/2} \exp\left\{q(t)\int |N(y)|^{-1/2} dy\right\}, \tag{31}$$

where $w = w(t)$ and $q = q(t)$ are two integral functions only depending on t. Thus, we can obtain the following solution of the vacuum Einstein's field equations in the coordinates (t, x, y, z)

$$ds^2 = (dt, dx, dy, dz)(g_{\mu\nu})(dt, dx, dy, dz)^T, \tag{32}$$

where

$$(g_{\mu\nu}) = \begin{pmatrix} 2K_t V & K_x V & K V_y & 0 \\ K_x V & 0 & 0 & 0 \\ K V_y & 0 & -K^2 & 0 \\ 0 & 0 & 0 & NK^2 \end{pmatrix}, \tag{33}$$

in which $N = N(y)$ is an arbitrary function of y, $K = K(t, x)$ is an arbitrary function of t, x, and V is given by (31).

By calculations, the Riemann curvature tensor reads

$$R_{\alpha\beta\mu\nu} = 0, \quad \forall\ \alpha\beta\mu\nu \neq 0202 \ \text{ or } \ 0303, \tag{34}$$

while

$$R_{0202} = Kwqq'|N|^{-1/2} \exp\left\{q\int |N|^{-1/2} dy\right\} \tag{35}$$

and

$$R_{0303} = Kwqq'|N|^{1/2} \exp\left\{q\int |N|^{-1/2} dy\right\}. \tag{36}$$

In order to analyze the singularities of the space-time (32), we introduce

Definition 1 *A point P in the space-time is called a geometric singular point, if there are some indexes $\alpha_0, \beta_0, \mu_0, \nu_0 \in \{0, 1, 2, 3\}$ such that*

$$R_{\alpha_0\beta_0\mu_0\nu_0} = \pm\infty \quad \text{but} \quad |\mathbf{R}| \triangleq |R^{\alpha\beta\gamma\delta} R_{\alpha\beta\gamma\delta}| < \infty \quad \text{at } P;$$

P is called a physical singular point, if it holds that

$$\mathbf{R} = \pm\infty \quad \text{at } P.$$

A low-dimensional sub-manifold Σ is called a geometric (resp. physical) singularity, if every point $(t, x, y, z) \in \Sigma$ is geometric (resp. physical) singular point. ∎

In Definition 1, the low-dimensional sub-manifold Σ might be a point, curve or surface in the space-time. On the other hand, physicists usually call "geometric singularities" defined in Definition 1 the coordinate singularities.

According to Definition 1, it is easy to check that, in the Schwarzschild space-time, the event horizon $r = 2M$ is geometric singularity, while the black hole $r = 0$ is physical singularity.

3 Time-periodic solutions

This section is devoted to constructing some new time-periodic solutions of the vacuum Einstein's field equations.

3.1 Regular time-periodic space-times with vanishing Riemann curvature tensor

Take $q = constant$ and let $V = \rho(t)\kappa(y)$, where κ is defined by

$$\kappa(y) = c_1\sqrt{|N|}\exp\left\{c_2\int |N|^{-1/2}dy\right\}, \tag{37}$$

in which c_1 and c_2 are two integrable constants. In this case, the solution to the vacuum Einstein's filed equations in the coordinates (t, x, y, z) reads

$$ds^2 = (dt, dx, dy, dz)(g_{\mu\nu})(dt, dx, dy, dz)^T, \tag{38}$$

where

$$(g_{\mu\nu}) = \begin{pmatrix} 2\rho\kappa\partial_t K & \rho\kappa\partial_x K & \rho K\partial_y\kappa & 0 \\ \rho\kappa\partial_x K & 0 & 0 & 0 \\ \rho K\partial_y\kappa & 0 & -K^2 & 0 \\ 0 & 0 & 0 & NK^2 \end{pmatrix}. \tag{39}$$

Theorem 1 *The vacuum Einstein's filed equations (2) have a solution described by (38) and (39), and the Riemann curvature tensor of this solution vanishes.* ■

As an example, let

$$\begin{cases} w(t) = \cos t, \\ q(t) = 0, \\ K(t,x) = e^x \sin t, \\ N(y) = -(2+\sin y)^2. \end{cases} \tag{40}$$

In the present situation, we obtain the following solution of the vacuum Einstein's filed equations (2)

$$(\eta_{\mu\nu}) = \begin{pmatrix} \eta_{00} & \eta_{01} & \eta_{02} & 0 \\ \eta_{01} & 0 & 0 & 0 \\ \eta_{02} & 0 & \eta_{22} & 0 \\ 0 & 0 & 0 & \eta_{33} \end{pmatrix}, \tag{41}$$

where

$$\begin{cases} \eta_{00} = 2e^x(2+\sin y)\cos^2 t, \\ \eta_{01} = \dfrac{1}{2}e^x(2+\sin y)\sin(2t), \\ \eta_{02} = \dfrac{1}{2}e^x\cos y\sin(2t), \\ \eta_{22} = -[e^x\sin t]^2, \\ \eta_{33} = -[e^x(2+\sin y)\sin t]^2. \end{cases} \tag{42}$$

By (4),

$$\eta \triangleq \det(\eta_{\mu\nu}) = -\frac{1}{4}e^{6x}(2+\sin y)^4\sin^4 t\sin^2(2t). \tag{43}$$

Property 1 The solution (41) of the vacuum Einstein's filed equations (2) is time-periodic. ■

Proof. In fact, the first equality in (42) implies that

$$\eta_{00} > 0 \quad \text{for } t \neq k\pi + \pi/2 \ \ (k \in \mathbb{N}) \ \text{ and } \ x \neq -\infty.$$

On the other hand, by direct calculations,

$$\begin{vmatrix} \eta_{00} & \eta_{01} \\ \eta_{01} & 0 \end{vmatrix} = -\frac{1}{4}e^{2x}(2+\sin y)^2\sin^2(2t) < 0,$$

$$\begin{vmatrix} \eta_{00} & \eta_{01} & \eta_{02} \\ \eta_{01} & 0 & 0 \\ \eta_{02} & 0 & \eta_{22} \end{vmatrix} = -\eta_{01}^2\eta_{22} > 0$$

and

$$\begin{vmatrix} \eta_{00} & \eta_{01} & \eta_{02} & 0 \\ \eta_{01} & 0 & 0 & 0 \\ \eta_{02} & 0 & \eta_{22} & 0 \\ 0 & 0 & 0 & \eta_{33} \end{vmatrix} = -\eta_{01}^2\eta_{22}\eta_{33} < 0$$

for $t \neq k\pi,\ k\pi + \pi/2$ $(k \in \mathbb{N})$ and $x \neq -\infty$.

In Property 3 below, we will show that $t = k\pi,\ k\pi + \pi/2$ $(k \in \mathbb{N})$ are the singularities of the space-time described by (41), but they are neither geometric singularities nor physical singularities, these non-essential singularities correspond to the event horizons of the space-time described by (41) with (42); while, when $x = -\infty$, the space-time (41) degenerates to a point.

The above discussion implies that the variable t is a time coordinate. Therefore, it follows from (42) that the Lorentzian metric

$$ds^2 = (dt, dx, dy, dz)(\eta_{\mu\nu})(dt, dx, dy, dz)^T \tag{44}$$

is indeed a time-periodic solution of the vacuum Einstein's field equations (2), where $(\eta_{\mu\nu})$ is given by (41). This proves Property 1. □

Noting (34)-(36) and the second equality in (40) gives

Property 2 The Lorentzian metric (44) (in which $(\eta_{\mu\nu})$ is given by (41) and (42)) describes a regular space-time, this space-time is Riemannian flat, that is to say, its Riemann curvature tensor vanishes. ■

Remark 1 *The first time-periodic solution to the vacuum Einstein's field equations was constructed by Kong and Liu [3]. The time-periodic solution presented in [3] also has the vanishing Riemann curvature tensor.*

It follows from (43) that the hypersurfaces $t = k\pi,\ k\pi + \pi/2 \quad (k \in \mathbb{N})$ and $x = \pm\infty$ are singularities of the space-time (44) (in which $(\eta_{\mu\nu})$ is given by (41) and (42)), however, by Property 2, these singularities are neither geometric singularities nor physical singularities. According to the definition of event horizon (see e.g., Wald [6]), it is easy to show that the hypersurfaces $t = k\pi,\ k\pi + \pi/2 \ (k \in \mathbb{N})$ and $x = +\infty$ are the event horizons of the space-time (44) (in which $(\eta_{\mu\nu})$ is given by (41) and (42)). Therefore, we have

Property 3 The Lorentzian metric (44) (in which $(\eta_{\mu\nu})$ is given by (41) and (42)) contains neither geometric nor physical singularities. These non-essential singularities consist of the hypersurfaces $t = k\pi,\ k\pi + \pi/2 \ (k \in \mathbb{N})$ and $x = \pm\infty$. The singularities $t = k\pi,\ k\pi + \pi/2 \ (k \in \mathbb{N})$ and $x = +\infty$ correspond to the event horizons, while, when $x = -\infty$, the space-time (44) degenerates to a point. ■

We now investigate the physical behavior of the space-time (44).

Fixing y and z, we get the induced metric

$$ds^2 = \eta_{00} dt^2 + 2\eta_{01} dt dx. \tag{45}$$

Consider the null curves in the (t, x)-plan, which are defined by

$$\eta_{00} dt^2 + 2\eta_{01} dt dx = 0. \tag{46}$$

Noting (42) gives

$$dt = 0 \quad \text{and} \quad \frac{dt}{dx} = -\tan t. \tag{47}$$

Thus, the null curves and light-cones are shown in Figure 1.

We next study the geometric behavior of the t-slices.

For any fixed $t \in \mathbb{R}$, it follows from (44) that the induced metric of the t-slice reads

$$ds^2 = \eta_{22} dy^2 + \eta_{33} dz^2 = -e^{2x} \sin^2 t [dy^2 + (2 + \sin y)^2 dz^2]. \tag{48}$$

When $t = k\pi \ (k \in \mathbb{N})$, the metric (48) becomes

$$ds^2 = 0.$$

This implies that the t-slice reduces to a point. On the other hand, in the present situation, the metric (44) becomes

$$ds^2 = 2e^x (2 + \sin y) dt^2.$$

When $t \neq k\pi \ (k \in \mathbb{N})$, (48) shows that the t-slice is a three-dimensional cone-like manifold centered at $x = -\infty$.

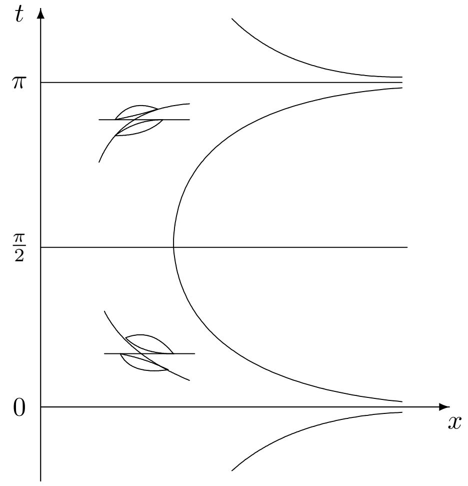

Figure 1: Null curves and light-cones in the domains $0 < t < \pi/2$ and $\pi/2 < t < \pi$.

3.2 Regular time-periodic space-times with non-vanishing Riemann curvature tensor

We next construct the regular time-periodic space-times with non-vanishing Riemann curvature tensor.

To do so, let

$$\begin{cases} w(t) = \cos t, \\ q(t) = \sin t, \\ K(x,t) = e^x \sin t, \\ N = -\dfrac{1}{(2+\sin y)^2}. \end{cases} \tag{49}$$

Then, by (31),

$$V = \frac{\cos t \exp\{(2y - \cos y)\sin t\}}{2 + \sin y}.$$

Thus, in the present situation, we have the following solution of the vacuum Einstein's field equations (2)

$$\widetilde{\eta}_{\mu\nu} = \begin{pmatrix} \widetilde{\eta}_{00} & \widetilde{\eta}_{01} & \widetilde{\eta}_{02} & 0 \\ \widetilde{\eta}_{01} & 0 & 0 & 0 \\ \widetilde{\eta}_{02} & 0 & \widetilde{\eta}_{22} & 0 \\ 0 & 0 & 0 & \widetilde{\eta}_{33} \end{pmatrix}, \tag{50}$$

where

$$\begin{cases} \widetilde{\eta}_{00} & = & \dfrac{2e^x \cos^2 t \exp\{(2y-\cos y)\sin t\}}{2+\sin y}, \\ \widetilde{\eta}_{01} & = & \dfrac{e^x \sin(2t) \exp\{(2y-\cos y)\sin t\}}{2(2+\sin y)}, \\ \widetilde{\eta}_{02} & = & e^{x+(2y-\cos y)\sin t}\left\{\sin t\cos t - \dfrac{\cos t\cos y}{(2+\sin y)^2}\right\}\sin t, \\ \widetilde{\eta}_{22} & = & -e^{2x}\sin^2 t, \\ \widetilde{\eta}_{33} & = & -\dfrac{e^{2x}\sin^2 t}{(2+\sin y)^2}. \end{cases} \tag{51}$$

By (4),

$$\widetilde{\eta} \stackrel{\triangle}{=} \det(\widetilde{\eta}_{\mu\nu}) = -(\widetilde{\eta}_{01})^2\widetilde{\eta}_{22}\widetilde{\eta}_{33} = -\frac{e^{6x+2(2y-\cos y)\sin t}\sin^2(2t)\sin^4 t}{4(2+\sin y)^4}. \tag{52}$$

Introduce

$$\triangle(t,x,y) = 6x + 2(2y-\cos y)\sin t.$$

Thus, it follows from (52) that

$$\widetilde{\eta} < 0 \tag{53}$$

for $t \neq k\pi,\ k\pi+\pi/2\ \ (k\in\mathbb{N})$ and $\triangle \neq -\infty$. It is obvious that the hypersurfaces $t = k\pi,\ k\pi+\pi/2\ \ (k\in\mathbb{N})$ and $\triangle = \pm\infty$ are the singularities of the space-time described by (50) with (51). As in Subsection 3.1, we can prove that the hypersurfaces $t = k\pi,\ k\pi+\pi/2\ \ (k\in\mathbb{N})$ are neither geometric singularities nor physical singularities, these non-essential singularities correspond to the event horizons of the space-time described by (50) with (51).

Similar to Property 1, we have

Property 4 The solution (50) (in which $(\widetilde{\eta}_{\mu\nu})$ is given by (51)) of the vacuum Einstein's filed equations (2) is time-periodic. ■

Similar to Property 2, we have

Property 5 The Lorentzian metric (50) (in which $(\widetilde{\eta}_{\mu\nu})$ is given by (51)) describes a regular space-time, this space-time has a non-vanishing Riemann curvature tensor. ■

Proof. In the present situation, by (34)

$$R_{\alpha\beta\mu\nu} = 0, \quad \forall\ \alpha\beta\mu\nu \neq 0202 \text{ or } 0303, \tag{54}$$

while

$$R_{0202} = e^{x+(2y-\cos y)\sin t}(2+\sin y)\cos^2 t\sin^2 t \tag{55}$$

and

$$R_{0303} = \frac{e^x\cos^2 t\sin^2 t\exp\{(2y-\cos y)\sin t\}}{2+\sin y}. \tag{56}$$

Property 5 follows from (54)-(56) directly. Thus the proof is completed. □

In particular, when $t \neq k\pi,\ k\pi+\pi/2\ \ (k\in\mathbb{N})$, it follows from (55) and (56) that

$$R_{0202},\ R_{0303} \longrightarrow \infty \quad \text{as } x+(2y-\cos y)\sin t \to \infty. \tag{57}$$

However, a direct calculation gives

$$\mathbf{R} \triangleq R^{\alpha\beta\gamma\delta} R_{\alpha\beta\gamma\delta} \equiv 0. \tag{58}$$

Thus, we obtain

Property 6 The Lorentzian metric (50) (in which $(\widetilde{\eta}_{\mu\nu})$ is given by (51)) contains neither geometric nor physical singularities. These non-essential singularities consist of the hypersurfaces $t = k\pi,\ k\pi + \pi/2\ \ (k \in \mathbb{N})$ and $\triangle = \pm\infty$, in which the hypersurfaces $t = k\pi,\ k\pi + \pi/2\ \ (k \in \mathbb{N})$ are the event horizons. Moreover, the Riemann curvature tensor satisfies the properties (57) and (58). ■

We next analyze the singularity behavior of $\triangle = \pm\infty$.

Case 1: Fixing $y \in \mathbb{R}$, we observe that

$$\triangle \to \pm\infty \Longleftrightarrow x \to \pm\infty.$$

This situation is similar to the case $x \to \pm\infty$ discussed in Subsection 3.1. That is to say, $x = +\infty$ corresponds to the event horizon, while, when $x \to -\infty$, the space-time (50) with (51) degenerates to a point.

Case 2: Fixing $x \in \mathbb{R}$, we observe that

$$\triangle \to \pm\infty \Longleftrightarrow y \to \pm\infty.$$

In the present situation, it holds that

$$t \neq k\pi \quad (k \in \mathbb{N}).$$

Without loss of generality, we may assume that

$$\sin t > 0.$$

For the case that $\sin t < 0$, we have a similar discussion. Thus, noting (57), we have

$$R_{0202},\ R_{0303} \longrightarrow \infty \quad \text{as}\ \ y \to \infty.$$

Moreover, by the definition of the event horizon we can show that $y = +\infty$ is not a event horizon. On the other hand, when $y \to -\infty$, the space-time (50) with (51) degenerates to a point.

Case 3: For the situation that $x \to \pm\infty$ and $y \to \pm\infty$ simultaneously, we have a similar discussion, here we omit the details.

For the space-time (50) with (51), the null curves and light-cones are shown just as in Figure 1. On the other hand, for any fixed $t \in \mathbb{R}$, the induced metric of the t-slice reads

$$ds^2 = \widetilde{\eta}_{22} dy^2 + \widetilde{\eta}_{33} dz^2 = -e^{2x} \sin^2 t [dy^2 + (2 + \sin y)^{-2} dz^2]. \tag{59}$$

Obviously, in the present situation, the t-slice possesses similar properties shown in the last paragraph in Subsection 3.1.

In particular, if we take (t, x, y, z) as the spherical coordinates (t, r, θ, φ) with $t \in \mathbb{R},\ r \in [0, \infty),\ \theta \in [0, 2\pi),\ \varphi \in [-\pi/2, \pi/2]$, then the metric (50) with (51) describes a regular time-periodic space-time with non-vanishing Riemann curvature tensor. This space-time does not contain any essential singularity, these non-essential singularities consist of the hypersurfaces $t = k\pi,\ k\pi + \pi/2\ \ (k \in \mathbb{N})$ which are the event horizons. The Riemann curvature tensor satisfies (58) and

$$R_{0202},\ R_{0303} \longrightarrow \infty \quad \text{as} \ \ r \to \infty.$$

Moreover, when $t \neq k\pi$ $(k \in \mathbb{N})$, the t-slice is a three dimensional bugle-like manifold with the base at $x = 0$; while, when $t = k\pi$ $(k \in \mathbb{N})$, the t-slice reduces to a point.

3.3 Time-periodic space-times with geometric singularities

This subsection is devoted to constructing the time-periodic space-times with physical singularities.

To do so, let

$$\begin{cases} w(t) = \cos t, \\ q(t) = \sin t, \\ K(x,t) = \dfrac{\sin t}{x^2}, \\ N = -\dfrac{1}{(2+\cos y)^2}. \end{cases} \tag{60}$$

Then, by (31) we have

$$V = \frac{\cos t \exp\{(2y + \sin y)\sin t)\}}{2 + \cos y}.$$

Thus, in the present situation, the solution of the vacuum Einstein's field equations (2) in the coordinates (t, x, y, z) reads

$$ds^2 = (dt, dx, dy, dz)(\hat{\eta}_{\mu\nu})(dt, dx, dy, dz)^T, \tag{61}$$

where

$$(\hat{\eta}_{\mu\nu}) = \begin{pmatrix} \hat{\eta}_{00} & \hat{\eta}_{01} & \hat{\eta}_{02} & 0 \\ \hat{\eta}_{01} & 0 & 0 & 0 \\ \hat{\eta}_{02} & 0 & \hat{\eta}_{22} & 0 \\ 0 & 0 & 0 & \hat{\eta}_{33} \end{pmatrix}, \tag{62}$$

in which

$$\begin{cases} \hat{\eta}_{00} &= \dfrac{2\cos^2 t \exp\{(\sin y + 2y)\sin t\}}{(2+\cos y)x^2}, \\ \hat{\eta}_{01} &= -\dfrac{\sin(2t)\exp\{(\sin y + 2y)\sin t\}}{(2+\cos y)x^3}, \\ \hat{\eta}_{02} &= \dfrac{\sin t}{x^2}\left\{\dfrac{\cos t \sin y}{(2+\cos y)^2} + \dfrac{\sin(2t)}{2}\right\}\exp\{(\sin y + 2y)\sin t\}, \\ \hat{\eta}_{22} &= -\dfrac{\sin^2 t}{x^4}, \\ \hat{\eta}_{33} &= -\dfrac{\sin^2 t}{(2+\cos y)^2 x^4}. \end{cases} \tag{63}$$

By (4), we have

$$\hat{\eta} \triangleq \det(\hat{\eta}_{\mu\nu}) = -(\hat{\eta}_{01})^2\hat{\eta}_{22}\hat{\eta}_{33} = -\frac{e^{2(2y+\sin y)\sin t}\sin^2(2t)\sin^4 t}{x^{14}(2+\cos y)^4}. \tag{64}$$

It follows from (63) that

$$\hat{\eta} < 0 \tag{65}$$

for $t \neq k\pi,\ k\pi+\pi/2\ \ (k \in \mathbb{N})$ and $x \neq 0$. Obviously, the hypersurfaces $t = k\pi,\ k\pi+\pi/2\ \ (k \in \mathbb{N})$ and $x = 0$ are the singularities of the space-time described by (61) with (62)-(63). As before, we can prove that the hypersurfaces $t = k\pi,\ k\pi + \pi/2\ \ (k \in \mathbb{N})$ are not essential (or, say, physical) singularities, and these non-essential singularities correspond to the event horizons of the space-time described by (61) with (62)-(63), however $x = 0$ is an essential (or, say, physical) singularity (see Property 8 below).

Similar to Property 1, we have

Property 7 The solution (61) (in which $(\hat{\eta}_{\mu\nu})$ is given by (62) and (63)) of the vacuum Einstein's field equations (2) is time-periodic. ■

Proof. In fact, the first equality in (63) implies that

$$\hat{\eta}_{00} > 0 \quad \text{for} \ \ t \neq k\pi + \pi/2 \quad (k \in \mathbb{N}) \ \text{ and } \ x \neq 0. \tag{66}$$

On the other hand, by direct calculations we have

$$\begin{vmatrix} \hat{\eta}_{00} & \hat{\eta}_{01} \\ \hat{\eta}_{01} & 0 \end{vmatrix} = -\hat{\eta}_{01}^2 < 0, \tag{67}$$

$$\begin{vmatrix} \hat{\eta}_{00} & \hat{\eta}_{01} & \hat{\eta}_{02} \\ \hat{\eta}_{01} & 0 & 0 \\ \hat{\eta}_{02} & 0 & \hat{\eta}_{22} \end{vmatrix} = -\hat{\eta}_{01}^2\hat{\eta}_{22} > 0 \tag{68}$$

and

$$\begin{vmatrix} \hat{\eta}_{00} & \hat{\eta}_{01} & \hat{\eta}_{02} & 0 \\ \hat{\eta}_{01} & 0 & 0 & 0 \\ \hat{\eta}_{02} & 0 & \hat{\eta}_{22} & 0 \\ 0 & 0 & 0 & \hat{\eta}_{33} \end{vmatrix} = -\hat{\eta}_{01}^2\hat{\eta}_{22}\hat{\eta}_{33} < 0 \tag{69}$$

for $t \neq k\pi,\ k\pi + \pi/2\ \ (k \in \mathbb{N})$ and $x \neq 0$.

The above discussion implies that the variable t is a time coordinate. Therefore, it follows from (63) that the Lorentzian metric (61) is indeed a time-periodic solution of the vacuum Einstein's field equations (2), where $(\hat{\eta}_{\mu\nu})$ is given by (63). This proves Property 7. □

Property 8 When $t \neq k\pi,\ k\pi + \pi/2\ \ (k \in \mathbb{N})$, for any fixed $y \in \mathbb{R}$ it holds that

$$R_{0202} \to +\infty \quad \text{and} \quad R_{0303} \to +\infty, \quad \text{as} \ \ x \to 0. \tag{70}$$

■

Proof. By direct calculations, we obtain from (35) and (36) that

$$R_{0202} = \frac{(2+\cos y)\sin^2(2t)\exp\{(\sin y + 2y)\sin t\}}{4x^2}, \tag{71}$$

and

$$R_{0303} = \frac{\sin^2(2t)\exp\{(\sin(y)+2y)\sin t\}}{4x^2(2+\cos y)}. \tag{72}$$

(70) follows from (71) and (72) directly. The proof is finished. □

On the other hand, a direct calculation yields

$$\mathbf{R} \triangleq R^{\alpha\beta\gamma\delta}R_{\alpha\beta\gamma\delta} \equiv 0. \tag{73}$$

Therefore, we have

Property 9 The Lorentzian metric (61) describes a time-periodic space-time, this space-time contains two kinds of singularities: the hypersurfaces $t = k\pi,\ k\pi + \pi/2\ \ (k \in \mathbb{N})$, which are neither geometric singularities nor physical singularities and correspond to the event horizons, and $x = 0$, which is a geometric singularity. ■

We now analyze the behavior of the singularities of the space-time characterized by (61) with (63).

By (64), we shall investigate the following cases: (a) $t = k\pi,\ k\pi + \pi/2\ \ (k \in \mathbb{N})$; (b) $y \to \pm\infty$; (c) $x \to \pm\infty$; (d) $x \to 0$.

Case a: $t = k\pi,\ k\pi + \pi/2\ \ (k \in \mathbb{N})$. According to the definition of the event horizon, the hypersurfaces $t = k\pi,\ k\pi + \pi/2\ (k \in \mathbb{N})$ are the event horizons of the space-time described by (61) with (63). On the other hand, by Definition 1, they are neither geometric singularities nor physical singularities.

Case b: $y \to \pm\infty$. Noting (64), in this case we may assume that $t \neq k\pi\ (k \in \mathbb{N})$ (if $t = k\pi$, then the situation becomes trivial). Without loss of generality, we may assume that $\sin t > 0$. Therefore, it follows from (71) and (72) that, for any fixed $x \neq 0$ it holds that

$$R_{0202},\ \ R_{0303} \longrightarrow \infty \quad \text{as}\ \ y \to +\infty \tag{74}$$

and

$$R_{0202},\ \ R_{0303} \longrightarrow 0 \quad \text{as}\ \ y \to -\infty. \tag{75}$$

(74) implies that $y = +\infty$ is also a geometric singularity, while $y = -\infty$ is not because of (75).

Case c: $x \to \pm\infty$. By (63), in this case the space-time characterized by (61) reduces to a point.

Case d: $x \to 0$. Property 8 shows that $x = 0$ is a geometric singularity. This is the biggest difference between the space-times presented in Subsections 3.1-3.2 and the one given this subsection. In order to illustrate its physical meaning, we take (t, x, y, z) as the spherical coordinates (t, r, θ, φ) with $t \in \mathbb{R},\ r \in [0, \infty),\ \theta \in [0, 2\pi),\ \varphi \in [-\pi/2, \pi/2]$. In the coordinates (t, r, θ, φ), the metric (61) with (63) describe a time-periodic space-time which possesses three kind of singularities:

(i) $t = k\pi,\ k\pi + \pi/2\ (k \in \mathbb{N})$: they are the event horizons, but neither geometric singularities nor physical singularities;

(ii) $r \to +\infty$: the space-time degenerates to a point;

(iii) $r \to 0$: it is a geometric singularity.

For the case (iii), in fact Property 8 shows that every point in the set

$$\mathfrak{S}_B \stackrel{\triangle}{=} \{(t,r,\theta,\varphi) \mid r=0,\ t \neq k\pi,\ k\pi+\pi/2\ (k\in\mathbb{N})\}$$

is a singular point. Noting (34) and (70), we name the set of singular points $\mathfrak{S}_B$ as a *geometric black hole*. Property 8 also shows that the space-time (61) is not homogenous and not asymptotically flat. This space-time perhaps has some new applications in cosmology due to the recent WMAP data, since the recent WMAP data show that our Universe exists anisotropy (see [2]). This inhomogenous property of the new space-time (61) may provide a way to give an explanation of this phenomena.

We next investigate the physical behavior of the space-time (61).

Fixing y and z, we get the induced metric

$$ds^2 = \hat{\eta}_{00}dt^2 + 2\hat{\eta}_{01}dtdx. \tag{76}$$

Consider the null curves in the (t,x)-plan defined by

$$\hat{\eta}_{00}dt^2 + 2\hat{\eta}_{01}dtdx = 0. \tag{77}$$

Noting (63) leads to

$$dt = 0 \quad \text{and} \quad \frac{dt}{dx} = -\frac{2\tan t}{x}. \tag{78}$$

Let

$$\rho = 2\ln|x|. \tag{79}$$

Then the second equation in (78) becomes

$$\frac{dt}{d\rho} = -\tan t. \tag{80}$$

Thus, in the (t,ρ)-plan the null curves and light-cones are shown in Figure 1 in which x should be replaced by ρ.

We now study the geometric behavior of the t-slices.

For any fixed $t\in\mathbb{R}$, the induced metric of the t-slice reads

$$ds^2 = -\frac{\sin^2 t}{x^4}[dy^2 + (2+\cos y)^{-2}dz^2]. \tag{81}$$

When $t = k\pi$ $(k\in\mathbb{N})$, the metric (81) becomes

$$ds^2 = 0.$$

This implies that the t-slice reduces to a point. On the other hand, in this case the metric (61) becomes

$$ds^2 = \frac{2}{(2+\cos y)x^2}dt^2.$$

When $t\neq k\pi$ $(k\in\mathbb{N})$, (81) shows that the t-slice is a three-dimensional manifold with cone-like singularities at $x=\infty$ and $x=-\infty$, respectively. In particular, if we take (t,x,y,z) as the spherical coordinates (t,r,θ,φ), then the induced metric (81) becomes

$$ds^2 = -\frac{\sin^2 t}{r^4}[d\theta^2 + (2+\cos\theta)^{-2}d\varphi^2]. \tag{82}$$

In this case the t-slice is a three-dimensional cone-like manifold centered at $r=\infty$.

At the end of this subsection, we would like to emphasize that the space-time (61) possesses a geometric singularity, i.e., $x=0$ which is named as a geometric black hole in this paper.

4 Summary and discussion

In this paper we describe a new method to find exact solutions of the vacuum Einstein's field equations (1). Using our method, we can construct many interesting exact solutions, in particular, the time-periodic solutions of the vacuum Einstein's field equations. More precisely, we have constructed three kinds of new time-periodic solutions of the vacuum Einstein's field equations: the regular time-periodic solution with vanishing Riemann curvature tensor, the regular time-periodic solution with finite Riemann curvature tensor and the time-periodic solution with geometric singularities. We have also analyzed the singularities of these new time-periodic solutions and investigate some new physical phenomena enjoyed by these new space-times. Here we would like to point out that, in this paper, when we discuss the time-periodic solutions, we use the time-coordinate t. In fact, we can also discuss the time-periodic solutions and singularities by means of a coordinate invariant way, in this way (and only this way), we need not introduce a special coordinate system.

In particular, in the spherical coordinates (t, r, θ, φ) we construct a time-periodic space-time with a geometric singularity. This space-time possesses an interesting and important singularity which is named as the geometric black hole. This space-time is inhomogenous and not asymptotically flat and can perhaps be used to explain the phenomenon that our Universe exists anisotropy from the recent WMAP data (see [2]). We believe some applications of these new space-times in modern cosmology and general relativity can be expected.

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