Long time behavior of smooth solutions to the compressible Euler equations with damping in several space variables

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Abstract

In this paper, we investigate the global existence and long time behavior of smooth solutions to the compressible Euler equations with damping in several space variables. Under suitable assumptions, we prove the existence and uniqueness of global smooth solution of the Cauchy problem and show that the solution and its derivatives decay to zero when t tends to the infinity.

Key words and phrases: compressible Euler equations, Cauchy problem, smooth solution, global existence, asymptotic behavior.

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1 Introduction

This paper concerns the global existence and long time behavior of smooth solution to the Cauchy problem for the compressible Euler equations

$$\begin{cases} \partial_t \rho + \nabla \cdot \rho v = 0, \\ \partial_t(\rho v) + \nabla \cdot (\rho v \otimes v) + \nabla P + a\rho v = 0 \end{cases} \tag{1.1}$$

with the initial data

$$t = 0 : \quad (\rho, v) = (\rho_0(x), v_0(x)), \tag{1.2}$$

where $\rho = \rho(t, x)$ and $v = (v_1, \cdots, v_n)^T$ denote the density and the velocity, respectively, a is a positive constant, and P, the pressure, is a function of ρ given by

$$P = P_0 + A\rho^\gamma, \tag{1.3}$$

in which P_0 is a positive constant, A and γ are two constants satisfying $A\gamma > 0$. For gas, γ stands for the adiabatic constant.

Compressible inviscid flow is governed by the Euler equations. The system (1.1) describes that the compressible gas flow passes a porous medium and the medium induces a friction force, proportional to the linear momentum in the opposite direction. It is a hyperbolic system. As a vacuum appears, it fails to be strictly hyperbolic. In general, the system (1.1) involves three mechanisms: nonlinear convection, lower-order dissipation of damping and the resonance due to the vacuum. In (1.3), when $\gamma > 1$, the gas is called the isentropic gas; when $\gamma = 1$, the gas is called the isothermal gas; when $\gamma = -1$, the gas is called the Chaplygin gas or von Karman-Tsien gas (see [13]).

For the existence, uniqueness and asymptotic behavior of global smooth solution to general one-dimensional hyperbolic systems, a complete result has been established (see [1], [18], [8], [4], [9] and [2]). For the one-dimensional Euler equations with damping, many interesting results have been obtained (see [3], [5]-[6], [11]-[12] and the references therein). For the multi-dimensional case, there have been several important results. For example, by detailed analysis on the Green function of the corresponding linearzied system and some energy estimates, Wang and Yang [15] proved the global existence of smooth solutions and gave the pointwise estimates of the solutions. For the three-dimensional case, Sideris, Thomases and Wang [14] proved the global existence and uniqueness of solution in $C([0, \infty); H^3) \bigcap C^1([0, \infty); H^2)$ and showed that the smooth solution converges to the constant background state in L^∞ at a rate of $(1 + t)^{-\frac{3}{2}}$ when t tends to the infinity.

In this paper, we first reduce the system (1.1) to a symmetric system which enjoys the property that the corresponding Cauchy problem for this symmetric system has a unique smooth solution if and only if the Cauchy problem (1.1)-(1.2) has a unique smooth solution. Based on this, we prove the global existence of solution of the Cauchy problem for the symmetric system in the space $C([0,\infty); H^m)\bigcap C^1([0,\infty); H^{m-1})$ $(n \le 4, m > \frac{n}{2}+1, m \in \mathbb{N})$, provided that the initial data is suitably small. Moreover, the asymptotic behavior of the solution is also investigated. These results cover and generalize the results presented in [14].

The method used in this paper improves one employed in [14]. Our approach is more effective for general dimension n and Sobolev space $H^m(\mathbb{R}^n)$. By establishing an important estimate, we obtain some decay estimates of the solution, these estimates did not obtain by [14] and [15].

Throughout this paper, we use the following notations: $L^p(\mathbb{R}^n)$ $(1 \le p \le \infty)$ denotes the usual space of all $L^p(\mathbb{R}^n)$-functions on $\mathbb{R}^n$ with norm

$$\|f\|_p = \|f\|_{L^p} \quad \text{and} \quad \|f\| = \|f\|_2,$$

H^m denotes the m-order Sobolev space on $\mathbb{R}^n$ with norm

$$\|f\|_{H^m} = \left(\|f\|^2 + \sum_{|\alpha|=m} \|D^\alpha f\|^2 \right)^{\frac{1}{2}},$$

where m is a positive integer.

The paper is organized as follows. In Section 2, we discuss the local solutions and state the local existence theorem on smooth solutions. Section 3 is devoted to establishing some *a priori* estimates which play an important role in this paper, while Section 4 is devoted to the global existence and uniqueness of smooth solution. In Section 5, under suitable assumptions we reprove the global existence of smooth solutions in a simple way and study the asymptotic behavior of the global smooth solutions.

2 Local solutions

This section is devoted to the local existence of smooth solutions to the compressible Euler equations with damping. To do so, we first reduce the system (1.1) to a symmetric system.

For the case: $A\gamma > 0$ and $\gamma \neq 1$, we introduce the sound speed

$$\varphi(\rho) = \sqrt{P'(\rho)} = \sqrt{A\gamma}\rho^{\frac{\gamma-1}{2}},$$

and set

$$\bar{\varphi} = \varphi(\bar{\rho})$$

which corresponds to the sound speed at a background density $\bar{\rho} > 0$. Let

$$u = \frac{2}{\gamma - 1}(\varphi(\rho) - \varphi(\bar{\rho})).$$

Then the system (1.1) can be reduced to

$$\begin{cases} \partial_t u + \bar{\varphi}\nabla \cdot v = -v \cdot \nabla u - \frac{\gamma-1}{2} u \nabla \cdot v, \\ \partial_t v + \bar{\varphi}\nabla u + av = -v \cdot \nabla v - \frac{\gamma-1}{2} u \nabla u, \end{cases} \tag{2.1}$$

and the initial data becomes

$$t = 0: \quad (u, v) = (u_0(x), v_0(x)), \tag{2.2}$$

where

$$u_0(x) = \frac{2}{\gamma - 1}(\varphi(\rho_0) - \varphi(\bar{\rho})).$$

For the case: $A > 0$ and $\gamma = 1$, let

$$\omega = \sqrt{A}(\ln \rho - \ln \bar{\rho}).$$

Then the system (1.1) can be reduced to

$$\begin{cases} \partial_t \omega + \bar{\varphi}\nabla \cdot v = -v \cdot \nabla \omega, \\ \partial_t v + \bar{\varphi}\nabla \omega + av = -v \cdot \nabla v, \end{cases} \tag{2.3}$$

while the initial data becomes

$$t = 0: \quad (\omega, v) = (\omega_0(x), v_0(x)), \tag{2.4}$$

where

$$\omega_0(x) = \sqrt{A}(\ln \rho_0 - \ln \bar{\rho}).$$

The proof of the following Lemma is straightforward.

Lemma 2.1 *Assume that $n \geq 1$. For any fixed $T > 0$, if $(\rho, v) \in C^1([0, T] \times \mathbb{R}^n)$ solves the Cauchy problem (1.1)-(1.2) with $\rho > 0$, then $(u, v) \in C^1([0, T] \times \mathbb{R}^n)$ solves the Cauchy problem (2.1)-(2.2) with $(\frac{\gamma-1}{2}u + \bar{\varphi}) > 0$. Conversely, if $(u, v) \in C^1([0, T] \times \mathbb{R}^n)$ solves the Cauchy problem (2.1)-(2.2) with $\varphi^{-1}(\frac{\gamma-1}{2}u + \bar{\varphi}) > 0$, then $(\rho, v) \in C^1([0, T] \times \mathbb{R}^n)$ solves the Cauchy problem (1.1)-(1.2) with $\rho > 0$.*

Lemma 2.2 *(I) If* $(u,v)\in C^1([0,T]\times\mathbb{R}^n)$ *is a uniformly bounded solution of the Cauchy problem (2.1)-(2.2) with* $\frac{\gamma-1}{2}u_0(x)+\bar{\varphi}>0$, *then*

$$\frac{\gamma-1}{2}u(x,t)+\bar{\varphi}>0 \quad \text{on} \quad [0,T]\times\mathbb{R}^n.$$

(II) If $(\rho,v)\in C^1([0,T]\times\mathbb{R}^n)$ *is a uniformly bounded solution of the Cauchy problem (1.1)-(1.2) with* $\rho_0(x)>0$, *then*

$$\rho(x,t)>0 \quad \text{on} \quad [0,T]\times\mathbb{R}^n.$$

Proof. Similar to the proof of Lemma 2.2 in [14], we can prove Lemma 2.2. We omit the proof.

Set

$$U(x,t)=(u(x,t),v(x,t))^T \quad \text{and} \quad U_0(x)=(u_0(x),v_0(x))^T.$$

Employing the method in [7] and [10], we can prove the following local existence theorem.

Theorem 2.1 *Assume that* $n\geq 1, m>\frac{n}{2}+1, m\in\mathbb{N}$ *and* $U_0(x)\in H^m(\mathbb{R}^n)$. *Then there exists a positive number* T_0 *such that the Cauchy problem (2.1)- (2.2) admits a unique solution* $U(x,t)\in C([0,T_0);H^m(\mathbb{R}^n))\bigcap C^1([0,T_0);H^{m-1}(\mathbb{R}^n))$.

For the case: $A>0$ and $\gamma=1$, similar to the above case, we have

Lemma 2.3 *Assume that* $n\geq 1$. *For any fixed* $T>0$, *if* $(\rho,v)\in C^1([0,T]\times\mathbb{R}^n)$ *solves the Cauchy problem (1.1)-(1.2) with* $\rho>0$, *then* $(\omega,v)\in C^1([0,T]\times\mathbb{R}^n)$ *solves the Cauchy problem (2.3)-(2.4) with* $\exp\left\{\frac{\omega}{\sqrt{A}}\right\}>0$. *Conversely, if* $(\omega,v)\in C^1([0,T]\times\mathbb{R}^n)$ *solves the Cauchy problem (2.3)-(2.4) with* $\exp(\frac{\omega}{\sqrt{A}})>0$, *then* $(\rho,v)\in C^1([0,T]\times\mathbb{R}^n)$ *solves the Cauchy problem (1.1)-(1.2) with* $\rho>0$.

Lemma 2.4 *(I) If* $(\omega,v)\in C^1([0,T]\times\mathbb{R}^n)$ *is a uniformly bounded solution of the Cauchy problem (2.3)-(2.4) with* $\exp(\frac{\omega_0(x)}{\sqrt{A}})>0$, *then*

$$\exp\left\{\frac{\omega(x,t)}{\sqrt{A}}\right\}>0 \quad \text{on} \quad [0,T]\times\mathbb{R}^n.$$

(II) If $(\rho,v)\in C^1([0,T]\times\mathbb{R}^n)$ *is a uniformly bounded solution of the Cauchy problem (1.1)-(1.2) with* $\rho_0(x)>0$, *then*

$$\rho(x,t)>0 \quad \text{on} \quad [0,T]\times\mathbb{R}^n.$$

Theorem 2.2 *Assume that* $n \geq 1, m > \frac{n}{2} + 1, m \in \mathbb{N}$ *and* $U_0(x) \in H^m(\mathbb{R}^n)$. *Then there exists a positive number* T_0 *such that the Cauchy problem (2.3)- (2.4) admits a unique solution* $U(x,t) \in C([0,T_0); H^m(\mathbb{R}^n)) \bigcap C^1([0,T_0); H^{m-1}(\mathbb{R}^n))$.

In the subsequent sections, we only consider the case that $A\gamma > 0$ and $\gamma \neq 1$. For the case that $A > 0$ and $\gamma = 1$, we have a similar discussion.

3 Some *a priori* estimates

In this section, we establish some *a priori* estimates, these estimates will play an important role in the proof of main results.

Lemma 3.1 *Suppose that* $1 \leq n \leq 4$, $m \in \mathbb{N}$ *and* $m > \frac{n}{2} + 1$. *Suppose furthermore that, for any given* $T > 0$, $U(x,t) \in C([0,T]; H^m(\mathbb{R}^n)) \bigcap C^1([0,T]; H^{m-1}(\mathbb{R}^n))$ *is a solution to the Cauchy problem (2.1)-(2.2). Then the following estimates hold*

$$\frac{d}{dt}\|U\|_{H^m}^2 + 2a\|v\|_{H^m}^2 \leq C(\|v\|\|\nabla U\|\|U\|_\infty + \|U\|_{H^m}\|D^m U\|^2), \tag{3.1}$$

$$\frac{d}{dt}\|\partial_t U\|_{H^{m-1}}^2 + 2a\|\partial_t v\|_{H^{m-1}}^2 \leq C\|\nabla U\|_{H^{m-1}}\|\partial_t U\|_{H^{m-1}}^2 \tag{3.2}$$

and

$$\begin{aligned}
&\frac{d}{dt}(\|U\|_{H^m}^2 + \|\partial_t U\|_{H^{m-1}}^2) + 2a(\|v\|_{H^m}^2 + \|\partial_t v\|_{H^{m-1}}^2) \\
&\quad \leq C(\|U\|_{H^m}^2 + \|\partial_t U\|_{H^{m-1}}^2)^{\frac{1}{2}}(\|v\|_{H^m}^2 + \|\partial_t v\|_{H^{m-1}}^2 + \|\nabla u\|_{H^{m-1}}^2 + \|\partial_t u\|_{H^{m-1}}^2),
\end{aligned} \tag{3.3}$$

where the positive constant C *depends on* γ, n *and* m, *independent of* T.

Proof. Multiplying the first equation of (2.1) by $2u$, the second one by $2v$ and summing them up gives

$$\frac{d}{dt}(|u|^2 + |v|^2) + 2a|v|^2 + 2\bar{\varphi}\nabla \cdot (uv) = -v \cdot \nabla(|u|^2 + |v|^2) - (\gamma - 1)u\nabla \cdot (uv). \tag{3.4}$$

Integrating (3.4) over $\mathbb{R}^n$ and using integration by parts leads to

$$\begin{aligned}
\frac{d}{dt}(\|u\|^2 + \|v\|^2) + 2a\|v\|^2 &= -\int_{\mathbb{R}^n} v \cdot \nabla(|u|^2 + |v|^2)dx + (\gamma - 1)\int_{\mathbb{R}^n} uv \cdot \nabla u dx \\
&\leq C\|v\|(\|\nabla(|u|^2 + |v|^2)\| + \|u\nabla u\|) \\
&\leq C\|v\|\|U\|_\infty\|\nabla U\|.
\end{aligned} \tag{3.5}$$

Taking α order derivatives of (2.1), multiplying the first equation by $2D^\alpha u$, the second one by $2D^\alpha v$ and then summing them up yields

$$\frac{d}{dt}(|D^\alpha u|^2 + |D^\alpha v|^2) + 2a|D^\alpha v|^2 + 2\bar{\varphi}\nabla\cdot(D^\alpha u D^\alpha v)$$
$$= -2(D^\alpha(v\cdot\nabla u)D^\alpha u + D^\alpha(v\cdot\nabla v)D^\alpha v) - (\gamma-1)(D^\alpha(u\nabla\cdot v)D^\alpha u + D^\alpha(u\nabla u)D^\alpha v). \tag{3.6}$$

Integrating (3.6) on $\mathbb{R}^n$ and using integration by parts gives

$$\begin{aligned}\frac{d}{dt}(\|D^\alpha u\|^2 + \|D^\alpha v\|^2) + 2a\|D^\alpha v\|^2 \\ = -2\int_{\mathbb{R}^n}(D^\alpha(v\cdot\nabla u)D^\alpha u + D^\alpha(v\cdot\nabla v)D^\alpha v)dx \\ -(\gamma-1)\int_{\mathbb{R}^n}(D^\alpha(u\nabla\cdot v)D^\alpha u + D^\alpha(u\nabla u)D^\alpha v)dx.\end{aligned} \tag{3.7}$$

Noting the following relations

$$v\cdot\nabla u = \sum_{i=1}^n v_i\frac{\partial u}{\partial x_i},\quad v\cdot\nabla v = \left(\sum_{i=1}^n v_i\frac{\partial v_1}{\partial x_i},\cdots,\sum_{i=1}^n v_i\frac{\partial v_n}{\partial x_i}\right)^T,\quad u\nabla\cdot v = \sum_{i=1}^n u\frac{\partial v_i}{\partial x_i},$$

$$u\nabla u = \left(u\frac{\partial u}{\partial x_1},\cdots,u\frac{\partial u}{\partial x_n}\right)^T,\quad D^\alpha(v\cdot\nabla v) = \left(\sum_{i=1}^n D^\alpha\left(v_i\frac{\partial v_1}{\partial x_i}\right),\cdots,\sum_{i=1}^n D^\alpha\left(v_i\frac{\partial v_n}{\partial x_i}\right)\right)^T$$

and

$$D^\alpha(u\nabla u) = \left(D^\alpha\left(u\frac{\partial u}{\partial x_1}\right),\cdots,D^\alpha\left(u\frac{\partial u}{\partial x_n}\right)\right)^T,$$

by Leibniz' formula we obtain

$$\begin{aligned} D^\alpha(v\cdot\nabla u) &= \sum_{i=1}^n D^\alpha(v_i\frac{\partial u}{\partial x_i}) = \sum_{i=1}^n\sum_{\beta\le\alpha}\begin{pmatrix}\alpha\\ \beta\end{pmatrix}D^\beta v_i D^{\alpha-\beta}\frac{\partial u}{\partial x_i} \\ &= \sum_{i=1}^n\Bigg[v_i D^\alpha\frac{\partial u}{\partial x_i} + \sum_{|\beta|=1}^{\beta\le\alpha}\begin{pmatrix}\alpha\\ \beta\end{pmatrix}D^\beta v_i D^{\alpha-\beta}\frac{\partial u}{\partial x_i} + \\ &\qquad \sum_{2\le|\beta|\le|\alpha|-1}^{\beta\le\alpha}\begin{pmatrix}\alpha\\ \beta\end{pmatrix}D^\beta v_i D^{\alpha-\beta}\frac{\partial u}{\partial x_i} + D^\alpha v_i\frac{\partial u}{\partial x_i}\Bigg],\end{aligned} \tag{3.8}$$

$$\begin{aligned} D^\alpha(v\cdot\nabla v)D^\alpha v &= \sum_{i=1}^n\sum_{j=1}^n\Bigg[v_i D^\alpha\frac{\partial v_j}{\partial x_i}D^\alpha v_j + \sum_{|\beta|=1}^{\beta\le\alpha}\begin{pmatrix}\alpha\\ \beta\end{pmatrix}D^\beta v_i D^{\alpha-\beta}\frac{\partial v_j}{\partial x_i}D^\alpha v_j + \\ &\qquad \sum_{2\le|\beta|\le|\alpha|-1}^{\beta\le\alpha}\begin{pmatrix}\alpha\\ \beta\end{pmatrix}D^\beta v_i D^{\alpha-\beta}\frac{\partial v_j}{\partial x_i}D^\alpha v_j + D^\alpha v_i\frac{\partial v_j}{\partial x_i}D^\alpha v_j\Bigg],\end{aligned} \tag{3.9}$$

$$
\begin{aligned}
D^\alpha(u\nabla\cdot v) &= \sum_{i=1}^n D^\alpha\left(u\frac{\partial v_i}{\partial x_i}\right) = \sum_{i=1}^n \left[uD^\alpha\frac{\partial v_i}{\partial x_i} + \sum_{|\beta|=1}^{\beta\le\alpha}\begin{pmatrix}\alpha\\ \beta\end{pmatrix} D^\beta u D^{\alpha-\beta}\frac{\partial v_i}{\partial x_i} + \right.\\
&\left.\sum_{2\le|\beta|\le|\alpha|-1}^{\beta\le\alpha}\begin{pmatrix}\alpha\\ \beta\end{pmatrix} D^\beta u D^{\alpha-\beta}\frac{\partial v_i}{\partial x_i} + D^\alpha u\frac{\partial v_i}{\partial x_i}\right]
\end{aligned}
\tag{3.10}
$$

and

$$
\begin{aligned}
D^\alpha(u\nabla u)\cdot D^\alpha v &= \sum_{i=1}^n D^\alpha\left(u\frac{\partial u}{\partial x_i}\right)D^\alpha v_i\\
&= \sum_{i=1}^n \left[uD^\alpha\frac{\partial u}{\partial x_i}D^\alpha v_i + \sum_{|\beta|=1}^{\beta\le\alpha}\begin{pmatrix}\alpha\\ \beta\end{pmatrix} D^\beta u D^{\alpha-\beta}\frac{\partial u}{\partial x_i}D^\alpha v_i + \right. \qquad (3.11)\\
&\left.\sum_{2\le|\beta|\le|\alpha|-1}^{\beta\le\alpha}\begin{pmatrix}\alpha\\ \beta\end{pmatrix} D^\beta u D^{\alpha-\beta}\frac{\partial u}{\partial x_i}D^\alpha v_i + D^\alpha u\frac{\partial u}{\partial x_i}D^\alpha v_i\right].
\end{aligned}
$$

Thus, it follows from (3.8)-(3.9) that

$$
\begin{aligned}
&-2\int_{\mathbb{R}^n}(D^\alpha(v\cdot\nabla u)D^\alpha u + D^\alpha(v\cdot\nabla v)D^\alpha v)dx\\
&= -2\sum_{i=1}^n\int_{\mathbb{R}^n} v_i\left(D^\alpha\frac{\partial u}{\partial x_i}D^\alpha u + \sum_{j=1}^n D^\alpha\frac{\partial v_j}{\partial x_i}D^\alpha v_j\right)dx\\
&-2\sum_{i=1}^n\sum_{|\beta|=1}^{\beta\le\alpha}\begin{pmatrix}\alpha\\ \beta\end{pmatrix}\int_{\mathbb{R}^n} D^\beta v_i\left(D^{\alpha-\beta}\frac{\partial u}{\partial x_i}D^\alpha u + \sum_{j=1}^n D^{\alpha-\beta}\frac{\partial v_j}{\partial x_i}D^\alpha v_j\right)dx\\
&-2\sum_{i=1}^n\sum_{2\le|\beta|\le|\alpha|-1}^{\beta\le\alpha}\begin{pmatrix}\alpha\\ \beta\end{pmatrix}\int_{\mathbb{R}^n} D^\beta v_i\left(D^{\alpha-\beta}\frac{\partial u}{\partial x_i}D^\alpha u + \sum_{j=1}^n D^{\alpha-\beta}\frac{\partial v_j}{\partial x_i}D^\alpha v_j\right)dx\\
&-2\sum_{i=1}^n\int_{\mathbb{R}^n} D^\alpha v_i\left(\frac{\partial u}{\partial x_i}D^\alpha u + \sum_{j=1}^n\frac{\partial v_j}{\partial x_i}D^\alpha v_j\right)dx\\
&\triangleq I_1 + I_2 + I_3 + I_4.
\end{aligned}
\tag{3.12}
$$

In what follows, we estimate I_i $(i = 1, 2, 3, 4)$.

Estimate of I_1: Using integration by parts, we have

$$
\begin{aligned}
I_1 &= -\sum_{i=1}^{n} \int_{\mathbb{R}^n} v_i \left(\frac{\partial}{\partial x_i} |D^\alpha u|^2 + \frac{\partial}{\partial x_i} |D^\alpha v|^2 \right) dx \\
&= -\int_{\mathbb{R}^n} v \cdot \nabla (|D^\alpha u|^2 + |D^\alpha v|^2) dx \\
&= \int_{\mathbb{R}^n} \nabla \cdot v (|D^\alpha u|^2 + |D^\alpha v|^2) dx \\
&\leq C \|\nabla \cdot v\|_\infty \|D^\alpha U\|^2.
\end{aligned} \tag{3.13}
$$

Estimate of I_2: Thanks to Hölder inequality and Minkowski inequality, we get

$$
\begin{aligned}
I_2 &\leq C \sum_{i=1}^{n} \sum_{|\beta|=1}^{\beta \leq \alpha} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} \|D^\beta v_i\|_\infty \left(\|D^{\alpha-\beta} \frac{\partial u}{\partial x_i}\| \|D^\alpha u\| + \sum_{j=1}^{n} \|D^{\alpha-\beta} \frac{\partial v_j}{\partial x_i}\| \|D^\alpha v_j\| \right) \\
&\leq C \sum_{i=1}^{n} \sum_{|\beta|=1}^{\beta \leq \alpha} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} \|D^\beta v_i\|_\infty \left(\|D^{\alpha-\beta} \frac{\partial u}{\partial x_i}\| \|D^\alpha u\| + \|D^{\alpha-\beta} \frac{\partial v}{\partial x_i}\| \|D^\alpha v\| \right) \\
&\leq C \|\nabla v\|_\infty \|D^\alpha U\|^2.
\end{aligned} \tag{3.14}
$$

Estimate of I_3: By Gagliardo- Nirenberg inequality, we obtain

$$
\|D^\beta v_i\|_4 \leq C \|v_i\|^{1-\theta_1} \|D^\alpha v_i\|^{\theta_1}, \tag{3.15}
$$

$$
\|D^{\alpha-\beta} \frac{\partial u}{\partial x_i}\|_4 \leq C \|u\|^{1-\theta_2} \|D^\alpha u\|^{\theta_2} \tag{3.16}
$$

and

$$
\|D^{\alpha-\beta} \frac{\partial v_j}{\partial x_i}\|_4 \leq C \|v_j\|^{1-\theta_2} \|D^\alpha v_j\|^{\theta_2}, \tag{3.17}
$$

where

$$
\theta_1 = \frac{|\beta| + \frac{n}{4}}{|\alpha|} \quad \text{and} \quad \theta_2 = \frac{|\alpha| - |\beta| + \frac{n}{4} + 1}{|\alpha|}.
$$

Then, using Hölder inequality, Minkowski inequality and (3.15)-(3.17), we have

$$\begin{aligned}
I_3 &\le C\sum_{i=1}^{n}\sum_{2\le|\beta|\le|\alpha|-1}^{\beta\le\alpha}\begin{pmatrix}\alpha\\ \beta\end{pmatrix}\|D^\beta v_i\|_4\left(\|D^{\alpha-\beta}\frac{\partial u}{\partial x_i}\|_4\|D^\alpha u\|+\sum_{j=1}^{n}\|D^{\alpha-\beta}\frac{\partial v_j}{\partial x_i}\|_4\|D^\alpha v_j\|\right)\\
&\le C\sum_{i=1}^{n}\sum_{2\le|\beta|\le|\alpha|-1}^{\beta\le\alpha}\begin{pmatrix}\alpha\\ \beta\end{pmatrix}\|v_i\|^{1-\theta_1}\|D^\alpha v_i\|^{\theta_1}\Big[\|u\|^{1-\theta_2}\|D^\alpha u\|^{\theta_2}\|D^\alpha u\|+\\
&\quad\sum_{j=1}^{n}\|v_j\|^{1-\theta_2}\|D^\alpha v_j\|^{\theta_2}\|D^\alpha v_j\|\Big]\\
&\le C\|U\|^{2-(\theta_1+\theta_2)}\|D^\alpha U\|^{\theta_1+\theta_2+1}.
\end{aligned} \tag{3.18}$$

Estimate of I_4: By Hölder inequality and Minkowski inequality,

$$\begin{aligned}
I_4 &\le C\sum_{i=1}^{n}\|D^\alpha v_i\|\left(\|\frac{\partial u}{\partial x_i}\|_\infty\|D^\alpha u\|+\sum_{j=1}^{n}\|\frac{\partial v_j}{\partial x_i}\|_\infty\|D^\alpha v_j\|\right)\\
&\le C\|\nabla U\|_\infty\|D^\alpha U\|^2.
\end{aligned} \tag{3.19}$$

Combining (3.12)-(3.14)and (3.18)-(3.19) leads to

$$-2\int_{\mathbb{R}^n}(D^\alpha(v\cdot\nabla u)D^\alpha u+D^\alpha(v\cdot\nabla v)D^\alpha v)dx\le C\|\nabla U\|_\infty\|D^\alpha U\|^2+C\|U\|^{2-(\theta_1+\theta_2)}\|D^\alpha U\|^{\theta_1+\theta_2+1}. \tag{3.20}$$

Similar to the proof of (3.20), we have

$$-(\gamma-1)\int_{\mathbb{R}^n}(D^\alpha(u\nabla\cdot v)D^\alpha u+D^\alpha(u\nabla u)D^\alpha v)dx\le C\|\nabla U\|_\infty\|D^\alpha U\|^2+C\|U\|^{2-(\theta_1+\theta_2)}\|D^\alpha U\|^{\theta_1+\theta_2+1}. \tag{3.21}$$

It follows from (3.7) and (3.20)-(3.21) that

$$\frac{d}{dt}(\|D^\alpha u\|^2+\|D^\alpha v\|^2)+2a\|D^\alpha v\|^2\le C\|\nabla U\|_\infty\|D^\alpha U\|^2+C\|U\|^{2-(\theta_1+\theta_2)}\|D^\alpha U\|^{\theta_1+\theta_2+1}. \tag{3.22}$$

Noting that $1<\theta_1+\theta_2<2$, from (3.5), (3.22) and Sobolev embedding theorem, we obtain (3.1) immediately.

In what follows, we prove (3.2).

Taking the ∂_t derivative of (2.1), multiplying the first equation of (2.1) by $2\partial_t u$, the second one by $2\partial_t v$ and then summing them up yields

$$\begin{aligned}
&\frac{d}{dt}(|\partial_t u|^2 + |\partial_t v|^2) + 2a|\partial_t v|^2 + 2\bar{\varphi}\nabla \cdot (\partial_t u \partial_t v) \\
&\quad = -2[\partial_t(v \cdot \nabla u)\partial_t u + \partial_t(v \cdot \nabla v) \cdot \partial_t v] - (\gamma - 1)[\partial_t(u\nabla \cdot v)\partial_t u + \partial_t(u\nabla u) \cdot \partial_t v] \\
&\quad = -2\partial_t v \cdot (\nabla u \partial_t u + \nabla v \partial_t v) - v \cdot \nabla(|\partial_t u|^2 + |\partial_t v|^2) \\
&\qquad -(\gamma - 1)\partial_t u(\nabla \cdot v\partial_t u + \nabla u \cdot \partial_t v) - (\gamma - 1)u\nabla \cdot (\partial_t v \partial_t u).
\end{aligned} \tag{3.23}$$

Integrating (3.23) over $\mathbb{R}^n$ and using integration by parts and Hölder inequality gives

$$\begin{aligned}
&\frac{d}{dt}(\|\partial_t u\|^2 + \|\partial_t v\|^2) + 2a\|\partial_t v\|^2 \\
&\quad = -2\int_{\mathbb{R}^n} \partial_t v \cdot (\nabla u \partial_t u + \nabla v \cdot \partial_t v)dx + \int_{\mathbb{R}^n} \nabla \cdot v(|\partial_t u|^2 + |\partial_t v|^2)dx - \\
&\qquad (\gamma - 1)\int_{\mathbb{R}^n} \partial_t u(\nabla \cdot v\partial_t u + \nabla u \cdot \partial_t v)dx + (\gamma - 1)\int_{\mathbb{R}^n} \nabla u \cdot \partial_t v \partial_t u dx \\
&\quad \le C\|\partial_t v\|_\infty(\|\nabla u\|\|\partial_t u\| + \|\nabla v\|\|\partial_t v\|) + C\|\nabla \cdot v\|_\infty(\|\partial_t u\|^2 + \|\partial_t v\|^2) + \\
&\qquad C\|\partial_t u\|_\infty(\|\nabla \cdot v\|\|\partial_t u\| + \|\nabla u\|\|\partial_t v\|) + C\|\nabla u\|_\infty\|\partial_t v\|\|\partial_t u\| \\
&\quad \le C\|\partial_t U\|_\infty\|\nabla U\|\|\partial_t U\| + C\|\nabla U\|_\infty\|\partial_t U\|^2.
\end{aligned} \tag{3.24}$$

Taking the $\partial_t D^\alpha$ derivative of (2.1), multiplying the first equation by $2\partial_t D^\alpha u$ and the second one by $2\partial_t D^\alpha v$, and then summing them up leads to

$$\begin{aligned}
&\frac{d}{dt}(|\partial_t D^\alpha u|^2 + |\partial_t D^\alpha v|^2) + 2a|\partial_t D^\alpha v|^2 + 2\bar{\varphi}\nabla \cdot (\partial_t D^\alpha u \partial_t D^\alpha v) \\
&\quad = -2[\partial_t D^\alpha(v \cdot \nabla u)\partial_t D^\alpha u + \partial_t D^\alpha(v \cdot \nabla v) \cdot \partial_t D^\alpha v] - \\
&\qquad (\gamma - 1)[\partial_t D^\alpha(u\nabla \cdot v)\partial_t D^\alpha u + \partial_t D^\alpha(u\nabla u) \cdot \partial_t D^\alpha v].
\end{aligned} \tag{3.25}$$

Integrating (3.25) over $\mathbb{R}^n$ and using integration by parts, we obtain

$$\begin{aligned}
&\frac{d}{dt}(\|\partial_t D^\alpha u\|^2 + \|\partial_t D^\alpha v\|^2) + 2a\|\partial_t D^\alpha v\|^2 \\
&\quad = -2\int_{\mathbb{R}^n} [D^\alpha(\partial_t v \cdot \nabla u)D^\alpha \partial_t u + D^\alpha(v \cdot \nabla \partial_t u)D^\alpha \partial_t u + \\
&\qquad D^\alpha(\partial_t v \cdot \nabla v) \cdot D^\alpha \partial_t v + D^\alpha(v \cdot \nabla \partial_t v) \cdot D^\alpha \partial_t v]dx \\
&\qquad -(\gamma - 1)\int_{\mathbb{R}^n} [D^\alpha(\partial_t u\nabla \cdot v)D^\alpha \partial_t u + D^\alpha(u\nabla \cdot \partial_t v)D^\alpha \partial_t u + \\
&\qquad D^\alpha(\partial_t u \nabla u) \cdot D^\alpha \partial_t v + D^\alpha(u\nabla \partial_t u) \cdot D^\alpha \partial_t v]dx.
\end{aligned} \tag{3.26}$$

Noting the following relations

$$\partial_t v \cdot \nabla u = \sum_{i=1}^n \partial_t v_i \frac{\partial u}{\partial x_i}, \quad v \cdot \nabla \partial_t u = \sum_{i=1}^n v_i \frac{\partial \partial_t u}{\partial x_i},$$

$$\partial_t v \cdot \nabla v = \left(\sum_{i=1}^n \partial_t v_i \frac{\partial v_1}{\partial x_i}, \cdots, \sum_{i=1}^n \partial_t v_i \frac{\partial v_n}{\partial x_i} \right)^T, \quad v \cdot \nabla \partial_t v = \left(\sum_{i=1}^n v_i \frac{\partial \partial_t v_1}{\partial x_i}, \cdots, \sum_{i=1}^n v_i \frac{\partial \partial_t v_n}{\partial x_i} \right)^T,$$

$$D^\alpha(\partial_t v \cdot \nabla v) = \left(\sum_{i=1}^n D^\alpha \left(\partial_t v_i \frac{\partial v_1}{\partial x_i} \right), \cdots, \sum_{i=1}^n D^\alpha \left(\partial_t v_i \frac{\partial v_n}{\partial x_i} \right) \right)^T,$$

$$D^\alpha(v \cdot \nabla \partial_t v) = \left(\sum_{i=1}^n D^\alpha \left(v_i \frac{\partial \partial_t v_1}{\partial x_i} \right), \cdots, \sum_{i=1}^n D^\alpha \left(v_i \frac{\partial \partial_t v_n}{\partial x_i} \right) \right)^T,$$

and using Leibniz' formula, we have

$$D^\alpha(\partial_t v \cdot \nabla u) = \sum_{i=1}^n D^\alpha \left(\partial_t v_i \frac{\partial u}{\partial x_i} \right) = \sum_{i=1}^n \left(\partial_t v_i D^\alpha \frac{\partial u}{\partial x_i} + \sum_{1 \le |\beta| \le |\alpha|-1}^{\beta \le \alpha} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} D^\beta \partial_t v_i D^{\alpha-\beta} \frac{\partial u}{\partial x_i} + D^\alpha \partial_t v_i \frac{\partial u}{\partial x_i} \right), \tag{3.27}$$

$$D^\alpha(v \cdot \nabla \partial_t u) = \sum_{i=1}^n D^\alpha \left(v_i \frac{\partial \partial_t u}{\partial x_i} \right) = \sum_{i=1}^n \left(v_i D^\alpha \frac{\partial \partial_t u}{\partial x_i} + \sum_{|\beta|=1}^{\beta \le \alpha} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} D^\beta v_i D^{\alpha-\beta} \frac{\partial \partial_t u}{\partial x_i} + \sum_{2 \le |\beta| \le |\alpha|}^{\beta \le \alpha} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} D^\beta v_i D^{\alpha-\beta} \frac{\partial \partial_t u}{\partial x_i} \right), \tag{3.28}$$

$$D^\alpha(\partial_t v \cdot \nabla v) \cdot D^\alpha \partial_t v = \sum_{i=1}^n \sum_{j=1}^n D^\alpha \left(\partial_t v_i \frac{\partial v_j}{\partial x_i} \right) D^\alpha \partial_t v_j = \sum_{i=1}^n \sum_{j=1}^n \left(\partial_t v_i D^\alpha \frac{\partial v_j}{\partial x_i} D^\alpha \partial_t v_j + \sum_{1 \le |\beta| \le |\alpha|-1}^{\beta \le \alpha} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} D^\beta \partial_t v_i D^{\alpha-\beta} \frac{\partial v_j}{\partial x_i} D^\alpha \partial_t v_j + D^\alpha \partial_t v_i \frac{\partial v_j}{\partial x_i} D^\alpha \partial_t v_j \right) \tag{3.29}$$

and

$$D^\alpha(v \cdot \nabla \partial_t v) \cdot D^\alpha \partial_t v = \sum_{i=1}^n \sum_{j=1}^n D^\alpha (v_i \frac{\partial \partial_t v_j}{\partial x_i}) D^\alpha \partial_t v_j = \sum_{i=1}^n \sum_{j=1}^n \left(v_i D^\alpha \frac{\partial \partial_t v_j}{\partial x_i} D^\alpha \partial_t v_j + \sum_{|\beta|=1}^{\beta \le \alpha} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} D^\beta v_i D^{\alpha-\beta} \frac{\partial \partial_t v_j}{\partial x_i} D^\alpha \partial_t v_j + \sum_{2 \le |\beta| \le |\alpha|}^{\beta \le \alpha} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} D^\beta v_i D^{\alpha-\beta} \frac{\partial \partial_t v_j}{\partial x_i} D^\alpha \partial_t v_j \right). \tag{3.30}$$

It follows from (3.27)-(3.30) that

$$
\begin{aligned}
&-2\int_{\mathbb{R}^n}[D^\alpha(\partial_t v\cdot\nabla u)D^\alpha\partial_t u+D^\alpha(v\cdot\nabla\partial_t u)D^\alpha\partial_t u+D^\alpha(\partial_t v\cdot\nabla v)\cdot D^\alpha\partial_t v+D^\alpha(v\cdot\nabla\partial_t v)\cdot D^\alpha\partial_t v]dx\\
&\quad=-2\sum_{i=1}^n\int_{\mathbb{R}^n}\partial_t v_i\left(D^\alpha\frac{\partial u}{\partial x_i}D^\alpha\partial_t u+\sum_{j=1}^n D^\alpha\frac{\partial v_j}{\partial x_i}D^\alpha\partial_t v_j\right)dx\\
&\quad\quad-2\sum_{i=1}^n\int_{\mathbb{R}^n}v_i\left(D^\alpha\frac{\partial\partial_t u}{\partial x_i}D^\alpha\partial_t u+\sum_{j=1}^n D^\alpha\frac{\partial\partial_t v_j}{\partial x_i}D^\alpha\partial_t v_j\right)dx\\
&\quad\quad-2\sum_{i=1}^n\sum_{|\beta|=1}^{\beta\le\alpha}\begin{pmatrix}\alpha\\ \beta\end{pmatrix}\int_{\mathbb{R}^n}D^\beta v_i\left(D^{\alpha-\beta}\frac{\partial\partial_t u}{\partial x_i}D^\alpha\partial_t u+\sum_{j=1}^n D^{\alpha-\beta}\frac{\partial\partial_t v_j}{\partial x_i}D^\alpha\partial_t v_j\right)dx\\
&\quad\quad-2\sum_{i=1}^n\sum_{1\le|\beta|\le|\alpha|-1}^{\beta\le\alpha}\begin{pmatrix}\alpha\\ \beta\end{pmatrix}\int_{\mathbb{R}^n}D^\beta\partial_t v_i\left(D^{\alpha-\beta}\frac{\partial u}{\partial x_i}D^\alpha\partial_t u+\sum_{j=1}^n D^{\alpha-\beta}\frac{\partial v_j}{\partial x_i}D^\alpha\partial_t v_j\right)dx\\
&\quad\quad-2\sum_{i=1}^n\sum_{2\le|\beta|\le|\alpha|}^{\beta\le\alpha}\begin{pmatrix}\alpha\\ \beta\end{pmatrix}\int_{\mathbb{R}^n}D^\beta v_i\left(D^{\alpha-\beta}\frac{\partial\partial_t u}{\partial x_i}D^\alpha\partial_t u+\sum_{j=1}^n D^{\alpha-\beta}\frac{\partial\partial_t v_j}{\partial x_i}D^\alpha\partial_t v_j\right)dx\\
&\quad\quad-2\sum_{i=1}^n\int_{\mathbb{R}^n}D^\alpha\partial_t v_i\left(\frac{\partial u}{\partial x_i}D^\alpha\partial_t u+\sum_{j=1}^n\frac{\partial v_j}{\partial x_i}D^\alpha\partial_t v_j\right)dx\\
&\quad\triangleq J_1+J_2+J_3+J_4+J_5+J_6.
\end{aligned}
\tag{3.31}
$$

In what follows, we estimate J_i $(i=1,\cdots,6)$.

Estimate of J_1 **:** Using Hölder inequality and Minkowski inequality, we have

$$
\begin{aligned}
J_1 &\le C\sum_{i=1}^n\|\partial_t v_i\|_\infty\left(\|D^\alpha\frac{\partial u}{\partial x_i}\|\|D^\alpha\partial_t u\|+\sum_{j=1}^n\|D^\alpha\frac{\partial v_j}{\partial x_i}\|\|D^\alpha\partial_t v_j\|\right)\\
&\le C\|\partial_t v\|_\infty(\|D^\alpha\nabla u\|\|D^\alpha\partial_t u\|+\|D^\alpha\nabla v\|\|D^\alpha\partial_t v\|)\\
&\le C\|\partial_t U\|_\infty\|D^\alpha\nabla U\|\|D^\alpha\partial_t U\|.
\end{aligned}
\tag{3.32}
$$

Estimate of J_2 **:** By integration by parts gives

$$
\begin{aligned}
J_2 &= -\sum_{i=1}^n\int_{\mathbb{R}^n}\left[v_i\frac{\partial}{\partial x_i}\left(|D^\alpha\partial_t u|^2+\sum_{j=1}^n|D^\alpha\partial_t v_j|^2\right)\right]dx\\
&= -\int_{\mathbb{R}^n}[v\cdot\nabla(|D^\alpha\partial_t u|^2+|D^\alpha\partial_t v|^2)]dx\\
&= \int_{\mathbb{R}^n}\nabla\cdot v(|D^\alpha\partial_t u|^2+|D^\alpha\partial_t v|^2)dx\le C\|\nabla U\|_\infty\|D^\alpha\partial_t U\|^2.
\end{aligned}
\tag{3.33}
$$

Estimate of J_3 : Using Hölder inequality and Minkowski inequality, we get

$$J_3 \leq C\|\nabla v\|_\infty(\|D^\alpha \partial_t u\|^2 + \|D^\alpha \partial_t v\|^2) \leq C\|\nabla U\|_\infty \|D^\alpha \partial_t U\|^2. \tag{3.34}$$

Estimate of J_4 : Thanks to Gagliardo-Nirenberg inequality,

$$\|D^\beta \partial_t v_i\|_4 \leq C\|\partial_t v_i\|^{1-\theta_1}\|D^\alpha \partial_t v_i\|^{\theta_1}, \tag{3.35}$$

$$\|D^{\alpha-\beta}\frac{\partial u}{\partial x_i}\|_4 \leq C\|\nabla u\|^{1-\theta_3}\|D^\alpha \nabla u\|^{\theta_3} \tag{3.36}$$

and

$$\|D^{\alpha-\beta}\frac{\partial v_j}{\partial x_i}\|_4 \leq C\|\nabla v_j\|^{1-\theta_3}\|D^\alpha \nabla v_j\|^{\theta_3}, \tag{3.37}$$

where

$$\theta_3 = \frac{|\alpha| - |\beta| + \frac{n}{4}}{|\alpha|}$$

and θ_1 is the same constant in (3.15).

Using Hölder inequality, Minkowski inequality and (3.35)-(3.37) , we obtain

$$\begin{aligned} J_4 &\leq C\sum_{i=1}^{n}\sum_{1\leq|\beta|\leq|\alpha|-1}^{\beta\leq\alpha}\begin{pmatrix}\alpha\\ \beta\end{pmatrix}\|D^\beta \partial_t v_i\|_4\left(\|D^{\alpha-\beta}\frac{\partial u}{\partial x_i}\|_4\|D^\alpha \partial_t u\| + \sum_{j=1}^{n}\|D^{\alpha-\beta}\frac{\partial v_j}{\partial x_i}\|_4\|D^\alpha \partial_t v_j\|\right) \\ &\leq C\sum_{i=1}^{n}\sum_{1\leq|\beta|\leq|\alpha|-1}^{\beta\leq\alpha}\begin{pmatrix}\alpha\\ \beta\end{pmatrix}\Big(\|\partial_t v_i\|^{1-\theta_1}\|D^\alpha \partial_t v_i\|^{\theta_1}\|\nabla u\|^{1-\theta_3}\|D^\alpha \nabla u\|^{\theta_3}\|D^\alpha \partial_t u\| + \\ &\qquad \sum_{j=1}^{n}\|\partial_t v_i\|^{1-\theta_1}\|D^\alpha \partial_t v_i\|^{\theta_1}\|\nabla v_j\|^{1-\theta_3}\|D^\alpha \nabla v_j\|^{\theta_3}\|D^\alpha \partial_t v_j\|\Big) \\ &\leq C\|\partial_t U\|^{1-\theta_1}\|D^\alpha \partial_t U\|^{1+\theta_1}\|\nabla U\|^{1-\theta_3}\|D^\alpha \nabla U\|^{\theta_3}. \end{aligned} \tag{3.38}$$

Estimate of J_5 : Using Hölder inequality and Minkowski inequality again yields

$$J_5 \leq C\sum_{i=1}^{n}\sum_{2\leq|\beta|\leq|\alpha|}^{\beta\leq\alpha}\begin{pmatrix}\alpha\\ \beta\end{pmatrix}\|D^\beta v_i\|_4\left(\|D^{\alpha-\beta}\frac{\partial \partial_t u}{\partial x_i}\|_4\|D^\alpha \partial_t u\| + \sum_{j=1}^{n}\|D^{\alpha-\beta}\frac{\partial \partial_t v_j}{\partial x_i}\|_4\|D^\alpha \partial_t v_j\|\right). \tag{3.39}$$

On the other hand, by Gagliardo-Nirenberg inequality, we have

$$\|D^\beta v_i\|_4 \leq C\|\nabla v_i\|^{1-\theta_4}\|D^\alpha \nabla v_i\|^{\theta_4}, \tag{3.40}$$

$$\|D^{\alpha-\beta}\frac{\partial \partial_t u}{\partial x_i}\|_4 \leq C\|\partial_t u\|^{1-\theta_2}\|D^\alpha \partial_t u\|^{\theta_2} \tag{3.41}$$

and

$$\|D^{\alpha-\beta}\frac{\partial \partial_t v_j}{\partial x_i}\|_4 \leq C\|\partial_t v_j\|^{1-\theta_2}\|D^\alpha \partial_t v_j\|^{\theta_2}, \tag{3.42}$$

where

$$\theta_4 = \frac{|\beta| + \frac{n}{4} - 1}{|\alpha|}$$

and θ_2 is the same constant in (3.16) and (3.17).

Thus, combining (3.39)-(3.42) leads to

$$\begin{aligned} J_5 &\le C\sum_{i=1}^{n}\sum_{2\le|\beta|\le|\alpha|}^{\beta\le\alpha}\begin{pmatrix}\alpha\\ \beta\end{pmatrix}\Big(\|\nabla v_i\|^{1-\theta_4}\|D^\alpha\nabla v_i\|^{\theta_4}\|\partial_t u\|^{1-\theta_2}\|D^\alpha\partial_t u\|^{\theta_2}\|D^\alpha\partial_t u\| \\ &\qquad + \sum_{j=1}^{n}\|\nabla v_i\|^{1-\theta_4}\|D^\alpha\nabla v_i\|^{\theta_4}\|\|\partial_t v_j\|^{1-\theta_2}\|D^\alpha\partial_t v_j\|^{\theta_2}\|D^\alpha\partial_t v_j\|\Big) \\ &\le C\|\nabla U\|^{1-\theta_4}\|D^\alpha\nabla U\|^{\theta_4}\|\partial_t U\|^{1-\theta_2}\|D^\alpha\partial_t U\|^{1+\theta_2}. \end{aligned} \tag{3.43}$$

Estimate of J_6 : By Hölder inequality and Minkowski inequality again,

$$J_6 \le C\sum_{i=1}^{n}\left[\|D^\alpha\partial_t v_i\|\left(\|\frac{\partial u}{\partial x_i}\|_\infty\|D^\alpha\partial_t u\| + \sum_{j=1}^{n}\|\frac{\partial v_j}{\partial x_i}\|_\infty\|D^\alpha\partial_t v_j\|\right)\right] \le C\|\nabla U\|_\infty\|D^\alpha\partial_t U\|^2. \tag{3.44}$$

Combining (3.31)-(3.34), (3.38) and (3.43)-(3.44) gives

$$\begin{aligned} &-2\int_{\mathbb{R}^n}[D^\alpha(\partial_t v\cdot\nabla u)D^\alpha\partial_t u + D^\alpha(v\cdot\nabla\partial_t u)D^\alpha\partial_t u \\ &\quad + D^\alpha(\partial_t v\cdot\nabla v)\cdot D^\alpha\partial_t v + D^\alpha(v\cdot\nabla\partial_t v)\cdot D^\alpha\partial_t v]dx \\ &\le C\|\partial_t U\|_\infty\|D^\alpha\nabla U\|\|D^\alpha\partial_t U\| + C\|\nabla U\|_\infty\|D^\alpha\partial_t U\|^2 + \\ &\quad C\|\partial_t U\|^{1-\theta_1}\|D^\alpha\partial_t U\|^{1+\theta_1}\|\nabla U\|^{1-\theta_3}\|D^\alpha\nabla U\|^{\theta_3} + \\ &\quad C\|\nabla U\|^{1-\theta_4}\|D^\alpha\nabla U\|^{\theta_4}\|\partial_t U\|^{1-\theta_2}\|D^\alpha\partial_t U\|^{1+\theta_2}. \end{aligned} \tag{3.45}$$

Similar to the proof of (3.45), we have

$$\begin{aligned} &-(\gamma-1)\int_{\mathbb{R}^n}[D^\alpha(\partial_t u\nabla\cdot v)D^\alpha\partial_t u + D^\alpha(u\nabla\cdot\partial_t v)D^\alpha\partial_t u \\ &\quad + D^\alpha(\partial_t u\nabla u)\cdot D^\alpha\partial_t v + D^\alpha(u\nabla\partial_t u)\cdot D^\alpha\partial_t v]dx \\ &\le C\|\partial_t U\|_\infty\|D^\alpha\nabla U\|\|D^\alpha\partial_t U\| + C\|\nabla U\|_\infty\|D^\alpha\partial_t U\|^2 + \\ &\quad C\|\partial_t U\|^{1-\theta_1}\|D^\alpha\partial_t U\|^{1+\theta_1}\|\nabla U\|^{1-\theta_3}\|D^\alpha\nabla U\|^{\theta_3} + \\ &\quad C\|\nabla U\|^{1-\theta_4}\|D^\alpha\nabla U\|^{\theta_4}\|\partial_t U\|^{1-\theta_2}\|D^\alpha\partial_t U\|^{1+\theta_2}. \end{aligned} \tag{3.46}$$

It follows from (3.26) and (3.45)-(3.46) that

$$
\begin{aligned}
&\frac{d}{dt}(\|\partial_t D^\alpha u\|^2+\|\partial_t D^\alpha v\|^2)+2a\|\partial_t D^\alpha v\|^2\\
&\quad\le C\|\partial_t U\|_\infty\|D^\alpha\nabla U\|\|D^\alpha\partial_t U\|+C\|\nabla U\|_\infty\|D^\alpha\partial_t U\|^2+\\
&\qquad C\|\partial_t U\|^{1-\theta_1}\|D^\alpha\partial_t U\|^{1+\theta_1}\|\nabla U\|^{1-\theta_3}\|D^\alpha\nabla U\|^{\theta_3}+\\
&\qquad C\|\nabla U\|^{1-\theta_4}\|D^\alpha\nabla U\|^{\theta_4}\|\partial_t U\|^{1-\theta_2}\|D^\alpha\partial_t U\|^{1+\theta_2}.
\end{aligned}
\tag{3.47}
$$

Using (3.24), (3.47) and Sobolev embedding theorem yields (3.2) immediately.

We now prove (3.3).

By (3.1) and (3.2), we have

$$
\begin{aligned}
&\frac{d}{dt}\left(\|U\|^2_{H^m}+\|\partial_t U\|^2_{H^{m-1}}\right)+2a\left(\|v\|^2_{H^m}+\|\partial_t v\|^2_{H^{m-1}}\right)\\
&\quad\le C\|v\|\|U\|_\infty\|\nabla U\|+C(\|\partial_t U\|_{H^{m-1}}+\|U\|_{H^m})(\|\nabla U\|^2_{H^{m-1}}+\|\partial_t U\|^2_{H^{m-1}}).
\end{aligned}
\tag{3.48}
$$

Thanks to the Sobolev embedding theorem, we obtain

$$
\|U\|_\infty\le C\|U\|_{H^{m-1}}\le C(\|U\|^2_{H^m}+\|\partial_t U\|^2_{H^{m-1}})^{\frac{1}{2}}.
$$

Therefore,

$$
\begin{aligned}
\|v\|\|U\|_\infty\|\nabla U\| &\le C(\|U\|^2_{H^m}+\|\partial_t U\|^2_{H^{m-1}})^{\frac{1}{2}}(\|v\|^2_{H^m}+\|\nabla U\|^2_{H^{m-1}})\\
&\le C(\|U\|^2_{H^m}+\|\partial_t U\|^2_{H^{m-1}})^{\frac{1}{2}}(\|v\|^2_{H^m}+\|\partial_t v\|^2_{H^{m-1}}+\|\nabla u\|^2_{H^{m-1}}+\|\partial_t u\|^2_{H^{m-1}}).
\end{aligned}
\tag{3.49}
$$

Noting that

$$
\|U\|_{H^m}+\|\partial_t U\|_{H^{m-1}}\le\sqrt{2}(\|U\|^2_{H^m}+\|\partial_t U\|^2_{H^{m-1}})^{\frac{1}{2}}.
$$

Thus, we have

$$
\begin{aligned}
&(\|U\|_{H^m}+\|\partial_t U\|_{H^{m-1}})(\|\nabla U\|^2_{H^{m-1}}+\|\partial_t U\|^2_{H^{m-1}})\\
&\quad\le\sqrt{2}(\|U\|^2_{H^m}+\|\partial_t U\|^2_{H^{m-1}})^{\frac{1}{2}}(\|v\|^2_{H^m}+\|\partial_t v\|^2_{H^{m-1}}+\|\nabla u\|^2_{H^{m-1}}+\|\partial_t u\|^2_{H^{m-1}}).
\end{aligned}
\tag{3.50}
$$

Combining (3.48)-(3.50) gives (3.3) immediately. Thus, the proof of Lemma 3.1 is completed. ■

Lemma 3.2 *Suppose that* $1\le n\le 4$, $m\in\mathbb{N}$ *and* $m>\frac{n}{2}+1$. *Suppose furthermore that, for any given* $T>0$, $U(x,t)\in C([0,T];H^m(\mathbb{R}^n))\bigcap C^1([0,T];H^{m-1}(\mathbb{R}^n))$ *is a solution of* *the Cauchy problem (2.1)-(2.2). Then, it holds that*

$$\begin{aligned}\|\nabla u\|_{H^{m-1}}^2+\|\partial_t u\|_{H^{m-1}}^2 \leq\ & C(\|u\|_{H^m}^2+\|\partial_t u\|_{H^{m-1}}^2)(\|\nabla u\|_{H^{m-1}}^2+\|\partial_t u\|_{H^{m-1}}^2)+\\ & C(\|v\|_{H^m}^2+\|\partial_t v\|_{H^{m-1}}^2)^2+C(\|v\|_{H^m}^2+\|\partial_t v\|_{H^{m-1}}^2)+\\ & C(\|u\|_{H^m}^2+\|\partial_t u\|_{H^{m-1}}^2)(\|v\|_{H^m}^2+\|\partial_t v\|_{H^{m-1}}^2).\end{aligned} \tag{3.51}$$

Where the positive constant C depends on $A, \bar{\rho}, \gamma, n$ and m, independent of T.

Proof. It follows from (2.1) that

$$\begin{cases} \partial_t u = -\bar{\varphi}\nabla\cdot v - v\cdot\nabla u - \frac{\gamma-1}{2}u\nabla\cdot v, \\ \nabla u = -\frac{1}{\bar{\varphi}}(\partial_t v + av + v\cdot\nabla v + \frac{\gamma-1}{2}u\nabla u). \end{cases} \tag{3.52}$$

By (3.52), we have

$$\begin{aligned}\|\nabla u\|_{H^{m-1}}^2+\|\partial_t u\|_{H^{m-1}}^2 \leq\ & C\left(\|v\|_{H^m}^2+\|\partial_t v\|_{H^{m-1}}^2+\|v\cdot\nabla u\|_{H^{m-1}}^2+\right.\\ & \left.\|u\nabla\cdot v\|_{H^{m-1}}^2+\|v\cdot\nabla v\|_{H^{m-1}}^2+\|u\nabla u\|_{H^{m-1}}^2\right).\end{aligned} \tag{3.53}$$

Using (3.8), (3.15)-(3.16) and Hölder inequality inequality, we get

$$\begin{aligned}\sum_{|\alpha|=m-1}\|D^\alpha(v\cdot\nabla u)\|^2 \leq\ & C\left[\|v\|_\infty^2\|u\|_{H^m}^2+\|\nabla v\|_\infty^2\|u\|_{H^{m-1}}^2+\right.\\ & \left.\|v\|_{H^{m-1}}^2\|u\|_{H^{m-1}}^2+\|\nabla u\|_\infty^2\|v\|_{H^{m-1}}^2\right].\end{aligned} \tag{3.54}$$

By the Sobolev embedding theorem, we obtain from (3.54) that

$$\sum_{|\alpha|=m-1}\|D^\alpha(v\cdot\nabla u)\|^2 \leq C(\|u\|_{H^m}^2+\|\partial_t u\|_{H^{m-1}}^2)(\|v\|_{H^m}^2+\|\partial_t v\|_{H^{m-1}}^2). \tag{3.55}$$

Thus,

$$\|v\cdot\nabla u\|_{H^{m-1}}^2 \leq C(\|u\|_{H^m}^2+\|\partial_t u\|_{H^{m-1}}^2)(\|v\|_{H^m}^2+\|\partial_t v\|_{H^{m-1}}^2). \tag{3.56}$$

Similarly, we can prove

$$\|u\nabla\cdot v\|_{H^{m-1}}^2 \leq C(\|u\|_{H^m}^2+\|\partial_t u\|_{H^{m-1}}^2)(\|v\|_{H^m}^2+\|\partial_t v\|_{H^{m-1}}^2), \tag{3.57}$$

$$\|v\cdot\nabla v\|_{H^{m-1}}^2 \leq C(\|v\|_{H^m}^2+\|\partial_t v\|_{H^{m-1}}^2)^2 \tag{3.58}$$

and

$$\|u\nabla u\|_{H^{m-1}}^2 \leq C(\|u\|_{H^m}^2+\|\partial_t u\|_{H^{m-1}}^2)(\|\nabla u\|_{H^{m-1}}^2+\|\partial_t u\|_{H^{m-1}}^2). \tag{3.59}$$

Combining (3.53) and (3.56)-(3.59) gives (3.51) immediately. Thus, the proof of Lemma 3.2 is completed. ■

4 Global existence and uniqueness

In this section, we will show that the Cauchy problem (2.1)-(2.2) admits a unique global smooth solution if the initial data is suitably small.

For the common positive constant C in (3.3) and (3.51), define

$$\delta_1 = \min\left\{1, \frac{1}{2C}, \frac{a^2}{C^2(4C+2)^2}\right\}. \tag{4.1}$$

Lemma 4.1 *Suppose that $1 \le n \le 4$, $m \in \mathbb{N}$ and $m > \frac{n}{2}+1$. Suppose furthermore that $U(x,t) \in C([0,T^0); H^m(\mathbb{R}^n)) \bigcap C^1([0,T^0); H^{m-1}(\mathbb{R}^n))$ is a solution to the Cauchy problem (2.1)-(2.2). Then it holds that*

$$\|U(t)\|_{H^m}^2 + \|\partial_t U(t)\|_{H^{m-1}}^2 \le C, \quad \forall\, t \in [0,T^0), \tag{4.2}$$

provided that the norm $\|U_0\|_{H^m}$ of the initial data is suitably small.

Proof. For the time being it is supposed that

$$\|U(t)\|_{H^m}^2 + \|\partial_t U(t)\|_{H^{m-1}}^2 \le \delta_1, \quad \forall\, t \in [0,T^0). \tag{4.3}$$

We shall explain that this hypothesis is reasonable later.

Noting (4.1), we obtain from Lemma 3.2 that

$$\|\nabla u(t)\|_{H^{m-1}}^2 + \|\partial_t u(t)\|_{H^{m-1}}^2 \le (4C+1)(\|v(t)\|_{H^m}^2 + \|\partial_t v(t)\|_{H^{m-1}}^2), \quad \forall\, t \in [0,T^0). \tag{4.4}$$

Thus, it follows from (3.3), (4.3)-(4.4) and (4.1) that

$$\frac{d}{dt}(\|U(t)\|_{H^m}^2 + \|\partial_t U(t)\|_{H^{m-1}}^2) + a(\|v(t)\|_{H^m}^2 + \|\partial_t v(t)\|_{H^{m-1}}^2) \le 0. \tag{4.5}$$

Using Gronwall inequality, we have

$$\|U(t)\|_{H^m}^2 + \|\partial_t U(t)\|_{H^{m-1}}^2 \le \|U(0)\|_{H^m}^2 + \|\partial_t U(0)\|_{H^{m-1}}^2, \quad \forall\, t \in [0,T^0). \tag{4.6}$$

By the system (2.1), we observe that

$$\|\partial_t U(t)\|_{H^{m-1}} \le C(\|U(t)\|_{H^m} + \|U(t)\|_{H^m}^2).$$

Hence, choosing $\|U_0\|_{H^m}$ suitably small, we have

$$\|U(0)\|_{H^m}^2 + \|\partial_t U(0)\|_{H^{m-1}}^2 \le C(\|U_0\|_{H^m}^2 + \|U_0\|_{H^m}^3 + \|U_0\|_{H^m}^4) \le \delta_1/2. \tag{4.7}$$

Combining (4.6) and (4.7) shows that the hypothesis (4.3) is reasonable. Thus, the proof of Lemma 4.1 is completed. ■

Combining Theorem 2.1 and Lemma 4.1 gives the following theorem.

Theorem 4.1 *Suppose that* $1 \le n \le 4$, $m \in \mathbb{N}$ *and* $m > \frac{n}{2} + 1$, $U_0 \in H^m(\mathbb{R}^n)$. *If* $\|U_0\|_{H^m}$ *is suitably small, then the Cauchy problem (2.1)-(2.2) admits a unique global smooth solution* $U = U(x,t) \in C([0,\infty); H^m(\mathbb{R}^n)) \bigcap C^1([0,\infty); H^{m-1}(\mathbb{R}^n))$.

5 Asymptotic behavior

In this section, we restudy the global existence of the smooth solution to the Cauchy problem (2.1)-(2.2) and investigate the asymptotic behavior of the solution. Throughout this section, we assume that the initial data $U_0(x)$ is in the space $L^1(\mathbb{R}^n) \bigcap H^m(\mathbb{R}^n)$, i.e., $U_0(x) \in L^1(\mathbb{R}^n) \bigcap H^m(\mathbb{R}^n)$ and the norm $\|U_0\|_1 + \|U_0\|_{H^m}$ is suitably small, where $m, n \in \mathbb{N}$ and satisfy that $2 \le n \le 4$, $m > \frac{n}{2} + 1$.

To do so, we consider the following linearized system of (2.1)

$$\begin{cases} \partial_t u + \nabla \cdot v = 0, \\ \partial_t v + \nabla u + av = 0, \end{cases} \tag{5.1}$$

where, without loss of generality, we take $\bar{\varphi} = 1$ in (2.1).

Similar to [14], using Fourier transform, we obtain from (5.1) that

$$\partial_t \hat{U}(\xi,t) = A(\xi)\hat{U}(\xi,t), \tag{5.2}$$

where $\hat{U}(\xi,t) = (\hat{u}(\xi,t), \hat{v}(\xi,t))^T$ and

$$A(\xi) = \begin{pmatrix} 0 & -i\xi \\ -i\xi^T & -aI_n \end{pmatrix},$$

in which T denotes the transpose of a row vector and I_n denotes the $n \times n$ unit matrix. The eigenvalues of $A(\xi)$ read

$$\lambda_1 = \cdots = \lambda_{n-1} = -a, \quad \lambda_n = -\frac{1}{2}(a + \sqrt{a^2 - 4|\xi|^2}), \quad \lambda_{n+1} = -\frac{1}{2}(a - \sqrt{a^2 - 4|\xi|^2}).$$

The eigenspace corresponding to the eigenvalue $-a$ is the subspace of vectors $(0,\zeta)$ with $\zeta \in \mathbb{R}^n$ and $\xi \cdot \zeta = 0$. The vector $h = (-i\lambda_{n+1}, \xi)^T$ is an eigenvector for the eigenvalue λ_n. Define the orthonormal set

$$\chi_i = (0, \zeta_i)^T \ \ (i = 1, \cdots, n-1), \quad \chi_n = \frac{h}{|h|},$$

where ζ_i $(i = 1, \cdots, n-1)$, ξ are mutually orthogonal row vectors in $\mathbb{R}^n$. We choose $\chi_{n+1} \in \mathbb{R}^{n+1}$ such that $\{\chi_i\}_{i=1}^{n+1}$ is an orthogonal basis of $\mathbb{R}^{n+1}$. Let $R(\xi)$ be the unitary matrix whose columns are $\chi_1, \cdots, \chi_{n+1}$. Then we have

$$A(\xi)R(\xi) = R(\xi)B(\xi),$$

where

$$B(\xi) = \begin{pmatrix} -a & 0 & \cdots & 0 & 0 & 0 \\ 0 & -a & \cdots & 0 & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & -a & 0 & 0 \\ 0 & 0 & \cdots & 0 & \lambda_n & z \\ 0 & 0 & \cdots & 0 & 0 & \lambda_{n+1} \end{pmatrix}_{(n+1)\times(n+1)},$$

in which

$$z = \begin{cases} -a, & a^2 - 4|\xi|^2 < 0, \\ 2\lambda_n, & a^2 - 4|\xi|^2 \geq 0. \end{cases}$$

We observe that

$$\hat{T}(t) = \begin{pmatrix} e^{-at} & 0 & \cdots & 0 & 0 & 0 \\ 0 & e^{-at} & \cdots & 0 & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & e^{-at} & 0 & 0 \\ 0 & 0 & \cdots & 0 & e^{\lambda_n t} & \frac{e^{\lambda_n t} - e^{\lambda_{n+1} t}}{\lambda_n - \lambda_{n+1}} z \\ 0 & 0 & \cdots & 0 & 0 & e^{\lambda_{n+1} t} \end{pmatrix}_{(n+1)\times(n+1)}$$

satisfies

$$\hat{T}'(\xi) = B(\xi)\hat{T}(t) \quad \text{and} \quad \hat{T}(0) = I,$$

so we have

$$\hat{T}(t) = \exp(tB(\xi)).$$

It follows that

$$\hat{W}(t) \overset{\triangle}{=} \exp(tA(\xi)) = R(\xi)\exp(tB(\xi))R^*(\xi) = R(\xi)\hat{T}(t)R^*(\xi).$$

Thus, the solution of the Cauchy problem (5.1), (2.2) is given by $U(x,t) = W(t)U_0$, where $W(t) = \mathbb{F}^{-1}\hat{W}(t)\mathbb{F}$ and $\mathbb{F}$ donates the Fourier transformation.

Lemma 5.1 *Assume that $m, n \in \mathbb{N}$, $2 \leq n \leq 4$, $m > \frac{n}{2} + 1$ and $U_0 \in L^1 \bigcap H^m(\mathbb{R}^n)$. Then it holds that*

$$\|W(t)U_0\|_\infty \leq C(1+t)^{-\frac{n}{2}}\|U_0\|_1 + Ce^{-\mu t}\|D^{[\frac{n}{2}]+1}U_0\|, \tag{5.3}$$

$$\|\nabla W(t)U_0\|_\infty \le C(1+t)^{-\frac{n+1}{2}}\|U_0\|_1 + Ce^{-\mu t}\|D^m U_0\|, \tag{5.4}$$

$$\|D^k W(t)U_0\| \le C(1+t)^{-\frac{n}{4}-\frac{k}{2}}\|U_0\|_1 + Ce^{-\mu t}\|D^k U_0\| \quad (k=0,1,\cdots,m), \tag{5.5}$$

$$\|W(t)U_0\|_{H^m} \le C\left[(1+t)^{-\frac{n}{4}} + (1+t)^{-\frac{n}{4}-\frac{m}{2}}\right]\|U_0\|_1 + Ce^{-\mu t}\|U_0\|_{H^m}, \tag{5.6}$$

where the positive constant C depends only on a and

$$\mu = \frac{a}{8}.$$

Proof. Similar to [14], we can show that the off-diagonal element of $\hat{T}(t)$ is bounded by

$$\mathfrak{F}(t) = \begin{cases} Ce^{-\frac{t|\xi|^2}{a}}, & \text{as } |\xi| < \nu = \frac{\sqrt{3}a}{4}, \\ Ce^{-\frac{at}{8}}, & \text{as } |\xi| \ge \nu = \frac{\sqrt{3}a}{4}. \end{cases} \tag{5.7}$$

A similar bound holds for the diagonal elements. Thus, $\hat{W}(t)$ is bounded by (5.7).

Using Hölder inequality and Hausdorff-Young inequality, we have

$$\begin{aligned} \|W(t)U_0\|_\infty &\le \|\hat{W}(t)\hat{U_0}\|_1 \le C\int_{|\xi|<\nu} e^{-\frac{t|\xi|^2}{a}}|\hat{U_0}|d\xi + C\int_{|\xi|\ge\nu} e^{-\mu t}|\hat{U_0}|d\xi \\ &\le C\|\hat{U_0}\|_\infty \int_{|\xi|<\nu} e^{-\frac{t|\xi|^2}{a}}d\xi + Ce^{-\mu t}\left(\int_{|\xi|\ge\nu}|\xi|^{-2([\frac{n}{2}]+1)}d\xi\right)^{\frac{1}{2}}\left(\int_{|\xi|\ge\nu}|\xi|^{2([\frac{n}{2}]+1)}|\hat{U_0}|^2 d\xi\right)^{\frac{1}{2}} \\ &\le C(1+t)^{-\frac{n}{2}}\|U_0\|_1 + Ce^{-\mu t}\|D^{[\frac{n}{2}]+1}U_0\|. \end{aligned}$$

This proves (5.3).

Using Hölder inequality and Hausdorff-Young inequality again and noting that fact that $2m-2>n$, we obtain

$$\begin{aligned} \|\nabla W(t)U_0\|_\infty &\le \|\hat{W}(t)|\xi|\hat{U_0}\|_1 \le C\int_{|\xi|<\nu} e^{-\frac{t|\xi|^2}{a}}|\xi||\hat{U_0}|d\xi + C\int_{|\xi|\ge\nu} e^{-\mu t}|\xi||\hat{U_0}|d\xi \\ &\le C\|\hat{U_0}\|_\infty \int_{|\xi|<\nu}|\xi|e^{-\frac{t|\xi|^2}{a}}d\xi + Ce^{-\mu t}\left(\int_{|\xi|\ge\nu}|\xi|^{2-2m}d\xi\right)^{\frac{1}{2}}\left(\int_{|\xi|\ge\nu}|\xi|^{2m}|\hat{U_0}|^2 d\xi\right)^{\frac{1}{2}} \\ &\le C(1+t)^{-\frac{n+1}{2}}\|U_0\|_1 + Ce^{-\mu t}\|D^m U_0\|. \end{aligned}$$

This proves (5.4).

Using Plancherel equality, for $k \in \{0,1,\cdots,m\}$ we get

$$\begin{aligned} \|D^k W(t)U_0\|^2 &= \|\hat{W}(t)|\xi|^k\hat{U_0}\|^2 \\ &\le C\int_{|\xi|<\nu}|\xi|^{2k}e^{-2\frac{t|\xi|^2}{a}}|\hat{U_0}|^2 d\xi + C\int_{|\xi|\ge\nu}|\xi|^{2k}e^{-2\mu t}|\hat{U_0}|^2 d\xi \\ &\le C(1+t)^{-\frac{n}{2}-k}\|\hat{U_0}\|_\infty^2 + Ce^{-2\mu t}\|D^k U_0\|^2 \\ &\le C(1+t)^{-\frac{n}{2}-k}\|U_0\|_1^2 + Ce^{-2\mu t}\|D^k U_0\|^2. \end{aligned}$$

This proves (5.5).

(5.6) follows (5.5) immediately. Thus, the proof of Lemma 5.1 is completed. ■

Lemma 5.2 *Suppose that $b > 1$ and $b \geq d > 0$. Then there is a positive constant C such that for all $t \geq 0$,*

$$J = \int_0^t (1+t-\tau)^{-b}(1+\tau)^{-d} d\tau \leq C(1+t)^{-d}.$$

See Zheng [17] for the proof of Lemma 5.2.

Theorem 5.1 *Suppose that $m, n \in \mathbb{N}$ and $2 \leq n \leq 4$, $m > \frac{n}{2}+1$. Then, there exists a $\delta > 0$ such that, for any $U_0 \in L^1(\mathbb{R}^n) \bigcap H^m(\mathbb{R}^n)$ satisfying*

$$\|U_0\|_1 + \|U_0\|_{H^m} \leq \delta, \tag{5.8}$$

the Cauchy problem (2.1)-(2.2) admits a unique solution $U = U(x,t) \in C([0,\infty); H^m(R^n)) \bigcap C^1([0,\infty); H^{m-1}(R^n))$. Moreover, the following estimate holds

$$\begin{aligned} \sup_{0\leq t<\infty} \Bigg\{ & (1+t)^{\frac{n}{2}} \|U\|_\infty + \sum_{k=0}^{[\frac{n}{2}]+1} (1+t)^{\frac{n}{4}+\frac{k}{2}} \|D^k U\| + \\ & \sum_{l=0}^{[\frac{n}{2}]} (1+t)^{\frac{n}{4}+\frac{l}{2}} \|D^l \partial_t U\| + \|U\|_{H^m} + \|\partial_t U\|_{H^{m-1}} \Bigg\} \leq C. \end{aligned} \tag{5.9}$$

Proof. It follows from the Duhamel principle that

$$U(x,t) = W(t)U_0 + \int_0^t W(t-\tau)F(U,\nabla U)(x,\tau)d\tau, \tag{5.10}$$

where

$$F(U,\nabla U) = \left(-v\cdot\nabla u - \frac{\gamma-1}{2}u\nabla\cdot v, -v\cdot\nabla v - \frac{\gamma-1}{2}u\nabla u\right)^T.$$

In what follows, we estimate the terms: $\|F(U,\nabla U)\|_1$, $\|F(U,\nabla U)\|$ and $\|D^k F(U,\nabla U)\|$, where $k = 1, \cdots, m$.

Using Minkowski inequality and Hölder inequality, we have

$$\|F(U,\nabla U)\|_1 \leq C\|U\|\|\nabla U\|. \tag{5.11}$$

Using Minkowski inequality again, we obtain

$$\|F(U,\nabla U)\| \leq C\|U\|_\infty \|\nabla U\|. \tag{5.12}$$

By Leibniz formula, Gagliardo-Nirenberg inequality and Hölder inequality, similar to the proof of (3.20), for $k \in \{1, \cdots, m\}$, we can prove

$$\|D^k F(U, \nabla U)\| \leq C\|U\|_{H^m}\|D^k U\|. \tag{5.13}$$

Let

$$N(t) = \sup_{0\leq\tau<t}\left\{(1+\tau)^{\frac{n}{2}}\|U(\tau)\|_\infty + \sum_{k=0}^{[\frac{n}{2}]+1}(1+\tau)^{\frac{n}{4}+\frac{k}{2}}\|D^k U(\tau)\| + \|U(\tau)\|_{H^m}\right\}. \tag{5.14}$$

It follows from Minkowski inequality, Lemma 5.1, (5.11), (5.13) and Lemma 5.2 that

$$\begin{aligned}
\|U(t)\|_\infty &\leq \|W(t)U_0\|_\infty + \int_0^t \|W(t-\tau)F(U,\nabla U)\|_\infty d\tau \\
&\leq C(1+t)^{-\frac{n}{2}}\|U_0\|_1 + Ce^{-\mu t}\|D^{[\frac{n}{2}]+1}U_0\| + \\
&\quad C\int_0^t (1+t-\tau)^{-\frac{n}{2}}\|F(U,\nabla U)\|_1 d\tau + C\int_0^t e^{-\mu(t-\tau)}\|D^{[\frac{n}{2}]+1}F(U,\nabla U)\|d\tau. \\
&\leq C(1+t)^{-\frac{n}{2}}(\|U_0\|_1 + \|U_0\|_{H^m}) + \\
&\quad CN^2(t)\int_0^t (1+t-\tau)^{-\frac{n}{2}}(1+\tau)^{-\frac{n+1}{2}}d\tau + CN^2(t)\int_0^t e^{-\mu(t-\tau)}(1+\tau)^{-(\frac{n}{4}+\frac{[\frac{n}{2}]+1}{2})}d\tau \\
&\leq C(1+t)^{-\frac{n}{2}}(\|U_0\|_1 + \|U_0\|_{H^m} + N^2(t)).
\end{aligned} \tag{5.15}$$

Noting that $k \leq [\frac{n}{2}]+1$, similar to (5.14), we have

$$\begin{aligned}
\|D^k U(t)\| &\leq \|D^k W(t)U_0\| + \int_0^t \|D^k W(t-\tau)F(U,\nabla U)\|d\tau \\
&\leq C(1+t)^{-(\frac{n}{4}+\frac{k}{2})}\|U_0\|_1 + Ce^{-\mu t}\|D^k U_0\| + \\
&\quad C\int_0^t (1+t-\tau)^{-(\frac{n}{4}+\frac{k}{2})}\|F(U,\nabla U)\|_1 d\tau + C\int_0^t e^{-\mu(t-\tau)}\|D^k F(U,\nabla U)\|d\tau \\
&\leq C(1+t)^{-(\frac{n}{4}+\frac{k}{2})}(\|U_0\|_1 + \|U_0\|_{H^m}) + \\
&\quad CN^2(t)\int_0^t (1+t-\tau)^{-(\frac{n}{4}+\frac{k}{2})}(1+\tau)^{-\frac{n+1}{2}}d\tau + CN^2(t)\int_0^t e^{-\mu(t-\tau)}(1+\tau)^{-(\frac{n}{4}+\frac{k}{2})}d\tau \\
&\leq C(1+t)^{-(\frac{n}{4}+\frac{k}{2})}(\|U_0\|_1 + \|U_0\|_{H^m} + N^2(t)).
\end{aligned} \tag{5.16}$$

Using Minkowski inequality, Lemma 5.1, (5.11)- (5.13) and Lemma 5.2 yields

$$\begin{aligned}\|U(t)\|_{H^m} &\leq \|W(t)U_0\|_{H^m} + \int_0^t \|W(t-\tau)F(U,\nabla U)\|_{H^m} d\tau \\ &\leq C((1+t)^{-\frac{n}{4}} + (1+t)^{-\frac{n}{4}-\frac{m}{2}})\|U_0\|_1 + Ce^{-\mu t}\|U_0\|_{H^m} + \\ &\quad C\int_0^t ((1+t-\tau)^{-\frac{n}{4}} + (1+t-\tau)^{-\frac{n}{4}-\frac{m}{2}})\|F(U,\nabla U)\|_1 d\tau + \\ &\quad C\int_0^t e^{-\mu(t-\tau)}\|F(U,\nabla U)\| d\tau + C\int_0^t e^{-\mu(t-\tau)}\|D^m F(U,\nabla U)\| d\tau \\ &\leq C(1+t)^{-\frac{n}{4}}(\|U_0\|_1 + \|U_0\|_{H^m}) + CN^2(t)\int_0^t (1+t-\tau)^{-\frac{n}{4}}(1+\tau)^{-\frac{n+1}{2}} d\tau + \\ &\quad CN^2(t)\int_0^t e^{-\mu(t-\tau)}(1+\tau)^{-(\frac{n}{4}+\frac{1}{2})} d\tau + CN^2(t)\int_0^t e^{-\mu(t-\tau)} d\tau \\ &\leq C(\|U_0\|_1 + \|U_0\|_{H^m} + N^2(t)).\end{aligned} \tag{5.17}$$

Thus, it follows from that (5.14)-(5.17) that

$$N(t) \leq C(\|U_0\|_1 + \|U_0\|_{H^m} + N^2(t)), \quad \forall\, t \geq 0. \tag{5.18}$$

Similar to [16], on the one hand, we can choose a positive real number ε which satisfies

$$\varepsilon > C\varepsilon^2, \tag{5.19}$$

where C is the absolute constant appearing in (5.18). On the other hand, choose $\delta > 0$ small enough such that, if

$$\|U_0\|_1 + \|U_0\|_{H^m} \leq \delta, \tag{5.20}$$

then $N(0) < \varepsilon$ and

$$\varepsilon > C(\|U_0\|_1 + \|U_0\|_{H^m}) + C\varepsilon^2. \tag{5.21}$$

Thus, the inequality (5.20) implies that

$$N(t) < \varepsilon \quad \text{for all } \; t \geq 0. \tag{5.22}$$

Otherwise, by the continuity of $N(t)$ and the fact that $N(0) < \varepsilon, N(t)$would necessarily be equal to ε at some $t > 0$, but in this case (5.18) would contradict (5.21). Therefore, $N(t)$ is bounded for all $t \geq 0$, i.e., there exists a positive constant $C > 0$ such that it holds that $N(t) \leq C$ for all $t > 0$.

Using (2.1) and $N(t) \le C$, we get

$$\|\partial_t U\|_{H^{m-1}} \le C(\|U\|_{H^m} + \|U\|_{H^m}^2) \le C. \tag{5.23}$$

Noting (2.1), (5.13) and using the fact that $N(t) \le C$, for $l \in \{0, \cdots, [\frac{n}{2}]\}$ we obtain

$$\begin{aligned} \|D^l \partial_t U\| &\le C\|D^{l+1}U\| + C\|D^l U\| + \|D^l F(U, \nabla U)\| \\ &\le C\|D^{l+1}U\| + C\|D^l U\| + C\|U\|_{H^m}\|D^l U\| \\ &\le C\left((1+t)^{-(\frac{n}{4}+\frac{l+1}{2})} + (1+t)^{-(\frac{n}{4}+\frac{l}{2})} + (1+t)^{-(\frac{n}{4}+\frac{l}{2})}\right) \\ &\le C(1+t)^{-(\frac{n}{4}+\frac{l}{2})}. \end{aligned} \tag{5.24}$$

Combining (5.23)-(5.24) and $N(t) \le C$ proves Theorem 5.1. Thus, the proof of Theorem 5.1 is completed. ■

Remark 5.1 *In Theorem 5.1, if* $2 \le n \le 3, m = 3$, *then from (5.4), (2.1) and the estimate* $\|D^3 F(U, \nabla U)\| \le C\|\nabla U\|_\infty \|D^2 U\|$, *we can prove that the solution to the Cauchy problem (2.1)-(2.2) also satisfies the following estimates*

$$\|\nabla U\|_\infty \le C(1+t)^{-\frac{n+1}{2}}$$

and

$$\|\partial_t U\|_\infty \le C(1+t)^{-\frac{n}{2}}.$$

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