

# New Results on the Geometry of the Moduli Space of Riemann Surfaces

*Dedicated to Professor LU QiKeng on the occasion of his 80th birthday*

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## Abstract

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## 1 Introduction

In this paper we briefly survey our recent results about the Mumford goodness of several canonical metrics on the moduli spaces of Riemann surfaces. We will also prove the dual Nakano negativity of the Weil-Petersson metric. As applications of these results we will deduce certain important results about the  $L^2$ -cohomology groups of the logarithmic tangent bundle over the compactified moduli spaces.

It is well known that, for an Hermitian vector bundle  $(E, h)$  over a closed manifold  $X$ , the Chern forms of  $h$  represent the Chern classes of  $E$ . However, this is no longer true if the base manifold is open. Mumford defined the goodness condition to study the currents of Chern forms of a singular Hermitian metric on a holomorphic bundle over a quasi-projective manifold in [1]. The goodness consists of growth conditions of the Hermitian metric, its connection form and curvature near the compactification divisor of the base manifold. The major property of a good metric is that the currents of its Chern forms define the Chern classes of this bundle. For details, please see Section 2.

In this note, we briefly describe our proof of the goodness of the Weil-Petersson metric, which was a long standing open problem. In fact we will present proofs of the goodness of the metrics induced by the Weil-Petersson metric, the Ricci and perturbed Ricci metrics on the logarithmic cotangent bundle over the Deligne-Mumford compactification of moduli space of Riemann surfaces (the DM moduli spaces). These works depend on our very accurate estimates of the asymptotics of the curvature and connection forms of these metrics in [2] and [3] together with the estimates of derivatives of the hyperbolic metric on Riemann surfaces. The computations

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and proofs are quite involved and very subtle. We will also present our proof of the dual Nakano negativity of the Weil-Petersson metric. In [4] Schumacher proved the strong negativity of the Weil-Petersson metric in the sense of Siu. Dual Nakano negativity is stronger than the strong negativity and several interesting consequences will follow. For example the goodness, together with the dual Nakano negativity of the Weil-Petersson metric, gives rise to interesting geometric consequences such as infinitesimal rigidity of the complex structure of the moduli spaces. We feel that our results open a new way to study the intersection theory on the moduli spaces by using differential geometric techniques, and also to apply index theory to the study of the geometry and topology of the moduli spaces and Teichmüller spaces.

We now describe the organization of this article. In Section 2 we recall some known works on the geometry of moduli spaces which include the degeneration of Riemann surfaces and hyperbolic metrics, the Ricci, perturbed Ricci and Kähler-Einstein metrics as well as their curvature properties, the asymptotic of Weil-Petersson metric, the Ricci and perturbed Ricci metrics as established in [2] and [3]. We also review Mumford's definition of good metrics.

In Section 3 we describe the main ideas of proving the goodness of the Weil-Petersson metric as well as the Ricci and the perturbed Ricci metrics. In Section 4 we describe the proof of the dual Nakano negativity of the Weil-Petersson metric. Finally, in Section 5 we apply these results to derive the vanishing theorem about certain cohomology groups and the infinitesimal rigidity of the moduli spaces. Our definition of the  $L^2$ -cohomology generalizes the usual one with trivial bundle coefficients. Here we have used both the Weil-Petersson metric and the Ricci metric and their goodness.

We remark that our previous results about the asymptotic behavior of the Kähler-Einstein metrics on the moduli spaces already imply an orbifold Chern number inequality for the logarithmic cotangent bundle which should give new information about positive divisors on the DM moduli spaces. In this paper, we will only give the main ideas and sketch the proofs of the results presented. For details and precise estimates, we refer the reader to [5] and [6].

This note is based on the first author's lecture at the International Conference in Honor of Professor Qi-Keng LU's Eightieth Birthday in June, 2006. Similar results have also been presented by the third author on the occasion of Professor Hironaka's eightieth birthday. The first author would like to thank Professor Qi-Keng LU for his guidance and help during the early stage of his research career.

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## 2 Curvature Formulae and Asymptotics

We first recall the deformation theory of Riemann surfaces as well as canonical metrics on the moduli spaces. Most of the results can be found in [2], [3], [7] and [1].

Let  $\mathcal{M}_{g,k}$  be the moduli space of Riemann surfaces of genus  $g$  with  $k$  punctures such that  $2g - 2 + k > 0$ . We know there is a unique Kähler-Einstein metric on such a Riemann surface. To clearly present the main ideas and to simplify the notation, throughout this paper, we will assume  $k = 0$  and  $g \geq 2$  and work on  $\mathcal{M}_g$ . Most of the results can be trivially generalized to  $\mathcal{M}_{g,k}$ .

By the Riemann-Roch theorem, we know that the complex dimension of the moduli space is

$n = \dim_{\mathbb{C}} \mathcal{M}_g = 3g - 3$ . Given a Riemann surface  $X$  of genus  $g \geq 2$ , we denote by  $\lambda$  the unique hyperbolic (Kähler-Einstein) metric on  $X$ . Let  $z$  be local holomorphic coordinate on  $X$ . We normalize  $\lambda$  such that

$$\partial_z \partial_{\bar{z}} \log \lambda = \lambda. \quad (2.1)$$

Let  $(s_1, \dots, s_n)$  be local holomorphic coordinates on  $\mathcal{M}_g$  near a point  $p$  and let  $X_s$  be the corresponding Riemann surfaces. Let  $\rho : T_s \mathcal{M}_g \rightarrow H^1(X_s, TX_s) \cong H^{0,1}(X_s, TX_s)$  be the Kodaira-Spencer map. Then the harmonic representative of  $\rho(\frac{\partial}{\partial s_i})$  is given by

$$\rho\left(\frac{\partial}{\partial s_i}\right) = \partial_{\bar{z}}(-\lambda^{-1} \partial_{s_i} \partial_{\bar{z}} \log \lambda) \frac{\partial}{\partial z} \otimes d\bar{z} = B_i. \quad (2.2)$$

By the Serre duality, we know

$$T_s^* \mathcal{M}_g \cong H^0(X_s, K_{X_s}^2)$$

where  $K_{X_s}$  is the canonical bundle of  $X_s$ . If we let  $a_i = -\lambda^{-1} \partial_{s_i} \partial_{\bar{z}} \log \lambda$  and let  $A_i = \partial_{\bar{z}} a_i$ , then the harmonic lift  $v_i$  of  $\frac{\partial}{\partial s_i}$  is given by

$$v_i = \frac{\partial}{\partial s_i} + a_i \frac{\partial}{\partial z}. \quad (2.3)$$

The well-known Weil-Petersson metric  $\omega_{WP} = \frac{\sqrt{-1}}{2} h_{i\bar{j}} ds_i \wedge d\bar{s}_j$  on  $\mathcal{M}_g$  is defined to be

$$h_{i\bar{j}}(s) = \int_{X_s} A_i \bar{A}_j dv \quad (2.4)$$

where  $dv = \frac{\sqrt{-1}}{2} \lambda dz \wedge d\bar{z}$  is the volume form on  $X_s$ . Correspondingly, the dual metric of the WP metric is given by

$$h^{i\bar{j}}(s) = \int_{X_s} \frac{\varphi_i \bar{\varphi}_j}{\lambda}$$

where  $\varphi_i \in H^0(X_s, K_{X_s}^2)$  is the holomorphic quadratic differential corresponding to the cotangent vector  $ds_i$ .

It was proved by Ahlfors that the Ricci curvature of the Weil-Petersson metric is negative. The upper bound of the Ricci curvature of the Weil-Petersson metric was conjectured by Royden and was proved by Wolpert [8].

In our work [2] we defined the Ricci metric  $\omega_\tau$ :

$$\omega_\tau = -\text{Ric}(\omega_{WP}) \quad (2.5)$$

and the perturbed Ricci metric  $\omega_{\bar{\tau}}$ :

$$\omega_{\bar{\tau}} = \omega_\tau + C\omega_{WP} \quad (2.6)$$

where  $C$  is a positive constant. These new Kähler metrics have good curvature and asymptotic properties and play important roles in our study.

Now we describe the curvature formulas of these metrics derived in [2] and [3]. We denote by  $f_{i\bar{j}} = A_i \bar{A}_j$  where  $A_i$  is the harmonic Beltrami differential corresponding to the local holomorphic vector field  $\frac{\partial}{\partial s_i}$ . It is clear that  $f_{i\bar{j}}$  is a function on  $X$ . We let  $\square = -\lambda^{-1} \partial_z \partial_{\bar{z}}$  be the

Laplace operator, let  $T = (\square + 1)^{-1}$  be the Green operator and let  $e_{i\bar{j}} = T(f_{i\bar{j}})$ . The functions  $e_{i\bar{j}}$  and  $f_{i\bar{j}}$  are building blocks of these curvature formulae.

The curvature formula of the Weil-Petersson metric was given by

$$R_{i\bar{j}k\bar{l}} = - \int_{X_s} (e_{i\bar{j}}f_{k\bar{l}} + e_{i\bar{l}}f_{k\bar{j}}) dv. \quad (2.7)$$

This formula was first established by Wolpert [8] and was generalized by Siu [9] and Schumacher [10] to higher dimensions. A short proof can be found in [2].

To describe the curvature formulae of the Ricci and perturbed Ricci metrics, we need to introduce several operators. These operators are compositions of the Maass operators. We recall the definitions here. Let  $X$  be a Riemann surface and let  $\kappa$  be its canonical bundle. For any integer  $p$ , let  $S(p)$  be the space of smooth sections of  $(\kappa \otimes \bar{\kappa}^{-1})^{\frac{p}{2}}$ . Fix a conformal metric  $ds^2 = \rho^2(z)|dz|^2$ .

**Definition 2.1.** *The Maass operators  $K_p$  and  $L_p$  are the metric derivatives  $K_p : S(p) \rightarrow S(p+1)$  and  $L_p : S(p) \rightarrow S(p-1)$  given by*

$$K_p(\sigma) = \rho^{p-1} \partial_z (\rho^{-p} \sigma)$$

and

$$L_p(\sigma) = \rho^{-p-1} \partial_{\bar{z}} (\rho^p \sigma)$$

where  $\sigma \in S(p)$ .

In the following, we will always use the Kähler-Einstein metric  $\lambda$  on  $X$  unless otherwise stated. We define the operator  $\xi_k : C^\infty(X_s) \rightarrow C^\infty(X_s)$  by

$$\xi_k(f) = \bar{\partial}^* (i(B_k)\partial f) = -\lambda^{-1} \partial_z (A_k \partial_z f) = -A_k K_1 K_0(f) \quad (2.8)$$

where  $K_0, K_1$  are the Maass operators defined above.

It was proved in [2] that  $\xi_k$  is the commutator of the Laplace operator and the Lie derivative in the direction  $v_k$ :

$$(\square + 1)v_k - v_k(\square + 1) = \square v_k - v_k \square = \xi_k. \quad (2.9)$$

We also need the commutator of the operator  $v_k$  and  $\bar{v}_l$ . In [2] we defined the operator  $Q_{k\bar{l}} : C^\infty(X_s) \rightarrow C^\infty(X_s)$  by

$$Q_{k\bar{l}}(f) = [\bar{v}_l, \xi_k](f) = \bar{P}(e_{k\bar{l}})P(f) - 2f_{k\bar{l}}\square f + \lambda^{-1} \partial_z f_{k\bar{l}} \partial_{\bar{z}} f \quad (2.10)$$

where  $P : C^\infty(X_s) \rightarrow \Gamma(\Lambda^{1,0}(T^{0,1}X_s))$  is the operator defined by

$$P(f) = \partial_z (\lambda^{-1} \partial_z f) = K_1 K_0(f).$$

The terms appeared in the curvature formulae of the Ricci and perturbed Ricci metrics are formally symmetric with respect to indices. For convenience, we recall the symmetrization operator defined in [2].

**Definition 2.2.** *Let  $U$  be any quantity which depends on indices  $i, k, \alpha, \bar{j}, \bar{l}, \bar{\beta}$ . The symmetrization operator  $\sigma_1$  is defined by taking the summation of all orders of the triple  $(i, k, \alpha)$ .*

Similarly,  $\sigma_2$  is the symmetrization operator of  $\bar{j}$  and  $\bar{\beta}$  and  $\tilde{\sigma}_1$  is the symmetrization operator of  $\bar{j}$ ,  $\bar{l}$  and  $\bar{\beta}$ .

Let  $\tilde{R}_{i\bar{j}k\bar{l}}$  and  $P_{i\bar{j}k\bar{l}}$  be the curvature tensors of the Ricci and perturbed Ricci metrics respectively. In [6] we established the following curvature formulae of these metrics:

$$\begin{aligned} \tilde{R}_{i\bar{j}k\bar{l}} = & -h^{\alpha\bar{\beta}} \left\{ \sigma_1 \sigma_2 \int_{X_s} \{ (T(\xi_k(e_{i\bar{j}}))\bar{\xi}_l(e_{\alpha\bar{\beta}}) + T(\xi_k(e_{i\bar{j}}))\bar{\xi}_\beta(e_{\alpha\bar{l}})) \} dv \right\} \\ & - h^{\alpha\bar{\beta}} \left\{ \sigma_1 \int_{X_s} Q_{k\bar{l}}(e_{i\bar{j}}) e_{\alpha\bar{\beta}} dv \right\} \\ & + \tau^{p\bar{q}} h^{\alpha\bar{\beta}} h^{\gamma\bar{\delta}} \left\{ \sigma_1 \int_{X_s} \xi_k(e_{i\bar{q}}) e_{\alpha\bar{\beta}} dv \right\} \left\{ \tilde{\sigma}_1 \int_{X_s} \bar{\xi}_l(e_{p\bar{j}}) e_{\gamma\bar{\delta}} dv \right\} \\ & + \tau_{p\bar{j}} h^{p\bar{q}} R_{i\bar{q}k\bar{l}} \end{aligned} \quad (2.11)$$

and

$$\begin{aligned} P_{i\bar{j}k\bar{l}} = & -h^{\alpha\bar{\beta}} \left\{ \sigma_1 \sigma_2 \int_{X_s} \{ T(\xi_k(e_{i\bar{j}}))\bar{\xi}_l(e_{\alpha\bar{\beta}}) + T(\xi_k(e_{i\bar{j}}))\bar{\xi}_\beta(e_{\alpha\bar{l}})) \} dv \right\} \\ & - h^{\alpha\bar{\beta}} \left\{ \sigma_1 \int_{X_s} Q_{k\bar{l}}(e_{i\bar{j}}) e_{\alpha\bar{\beta}} dv \right\} \\ & + \tilde{\tau}^{p\bar{q}} h^{\alpha\bar{\beta}} h^{\gamma\bar{\delta}} \left\{ \sigma_1 \int_{X_s} \xi_k(e_{i\bar{q}}) e_{\alpha\bar{\beta}} dv \right\} \left\{ \tilde{\sigma}_1 \int_{X_s} \bar{\xi}_l(e_{p\bar{j}}) e_{\gamma\bar{\delta}} dv \right\} \\ & + \tau_{p\bar{j}} h^{p\bar{q}} R_{i\bar{q}k\bar{l}} + CR_{i\bar{j}k\bar{l}}. \end{aligned} \quad (2.12)$$

From the curvature formula (2.7), it is easy to get information of the sign of the curvature of the Weil-Petersson metric by using integration by part and maximum principle. However, the curvature formulae of the Ricci and perturbed Ricci metrics are too complicated to use directly. Thus we need to look at the asymptotic behavior of these metrics.

We now recall geometric construction of the DM moduli space which is due to Earle-Marden and the degeneration of hyperbolic metrics. Please see [2] and [8] for details.

Let  $\overline{\mathcal{M}}_g$  be the Deligne-Mumford compactification of  $\mathcal{M}_g$  and let  $D = \overline{\mathcal{M}}_g \setminus \mathcal{M}_g$ . It was shown in [11] that  $D$  is a divisor with only normal crossings. A point  $y \in D$  corresponds to a stable nodal surface  $X_y$ . A point  $p \in X_y$  is a node if there is a neighborhood of  $p$  which is isometric to the germ  $\{(u, v) \mid uv = 0, |u|, |v| < 1\} \subset \mathbb{C}^2$ . Let  $p_1, \dots, p_m \in X_y$  be the nodes.  $X_y$  is stable if each connected component of  $X_y \setminus \{p_1, \dots, p_m\}$  has negative Euler characteristic.

Fix a point  $y \in D$ , we assume the corresponding Riemann surface  $X_y$  has  $m$  nodes. Now for any point  $s \in \mathcal{M}_g$  lying in a neighborhood of  $y$ , the corresponding Riemann surface  $X_s$  can be decomposed into the thin part which is a disjoint union of  $m$  collars and the thick part where the injectivity radius with respect to the Kähler-Einstein metric is uniformly bounded from below.

There are two kinds of local holomorphic coordinate on a collar or near a node. We first recall the rs-coordinate defined by Wolpert in [7]. In the node case, given a nodal surface  $X$  with a node  $p \in X$ , we let  $a, b$  be two punctures which are glued together to form  $p$ .

**Definition 2.3.** *A local coordinate chart  $(U, u)$  near  $a$  is called rs-coordinate if  $u(a) = 0$  where  $u$  maps  $U$  to the punctured disc  $0 < |u| < c$  with  $c > 0$ , and the restriction to  $U$  of the Kähler-Einstein metric on  $X$  can be written as  $\frac{1}{2|u|^2(\log|u|)^2} |du|^2$ . The rs-coordinate  $(V, v)$  near  $b$  is defined in a similar way.*

In the collar case, given a closed surface  $X$ , we assume there is a closed geodesic  $\gamma \subset X$  such that its length  $l = l(\gamma) < c_*$  where  $c_*$  is the collar constant.

**Definition 2.4.** A local coordinate chart  $(U, z)$  is called *rs-coordinate* at  $\gamma$  if  $\gamma \subset U$  where  $z$  maps  $U$  to the annulus  $c^{-1}|t|^{\frac{1}{2}} < |z| < c|t|^{\frac{1}{2}}$ , and the Kähler-Einstein metric on  $X$  can be written as  $\frac{1}{2}(\frac{\pi}{\log|t|} \frac{1}{|z|} \csc \frac{\pi \log|z|}{\log|t|})^2 |dz|^2$ .

The existence of collar was due to Keen [12]. We formulate this theorem in the following:

**Lemma 2.1.** Let  $X$  be a closed surface and let  $\gamma$  be a closed geodesic on  $X$  such that the length  $l$  of  $\gamma$  satisfies  $l < c_*$ . Then there is a collar  $\Omega$  on  $X$  with holomorphic coordinate  $z$  defined on  $\Omega$  such that

- (1)  $z$  maps  $\Omega$  to the annulus  $\{\frac{1}{c}e^{-\frac{2\pi^2}{l}} < |z| < c\}$  for  $c > 0$ ;
- (2) the Kähler-Einstein metric on  $X$  restricted to  $\Omega$  is given by

$$\left(\frac{1}{2}u^2r^{-2} \csc^2 \tau\right) |dz|^2 \quad (2.13)$$

where  $u = \frac{l}{2\pi}$ ,  $r = |z|$  and  $\tau = u \log r$ ;

- (3) the geodesic  $\gamma$  is given by the equation  $|z| = e^{-\frac{\pi^2}{l}}$ ;

(4) the constant  $c$  has a lower bound such that the area of  $\Omega$  is bounded from below by a universal constant.

We call such a collar  $\Omega$  a *genuine collar*.

Now we describe the pinching coordinate chart of  $\overline{\mathcal{M}}_g$  near the divisor  $D$  [7]. Let  $X_0$  be a nodal surface corresponding to a codimension  $m$  boundary point and let  $p_1, \dots, p_m$  be the nodes of  $X_0$ . Then  $\tilde{X}_0 = X_0 \setminus \{p_1, \dots, p_m\}$  is a union of punctured Riemann surfaces. Fix  $rs$ -coordinate charts  $(U_i, \eta_i)$  and  $(V_i, \zeta_i)$  at  $p_i$  for  $i = 1, \dots, m$  such that all the  $U_i$  and  $V_i$  are mutually disjoint. Now pick an open set  $U_0 \subset \tilde{X}_0$  such that the intersection of each connected component of  $\tilde{X}_0$  and  $U_0$  is a nonempty relatively compact set and the intersection  $U_0 \cap (U_i \cup V_i)$  is empty for all  $i$ . Now pick Beltrami differentials  $\nu_{m+1}, \dots, \nu_n$  which are supported in  $U_0$  and span the tangent space at  $\tilde{X}_0$  of the deformation space of  $\tilde{X}_0$ . Let  $\Delta_\varepsilon^{n-m} \subset \mathbb{C}^{n-m}$  be the polydisc of radius  $\varepsilon$ . For  $t'' = (t_{m+1}, \dots, t_n) \in \Delta_\varepsilon^{n-m}$ , let  $\nu(t'') = \sum_{i=m+1}^n t_i \nu_i$ . We assume  $|t''| = (\sum_{i=m+1}^n |t_i|^2)^{\frac{1}{2}}$  small enough such that  $|\nu(t'')| < 1$ . The nodal surface  $X_{0,t''}$  is obtained by solving the Beltrami equation  $\bar{\partial}w = \nu(t'')\partial w$ . Since  $\nu(t'')$  is supported in  $U_0$ ,  $(U_i, \eta_i)$  and  $(V_i, \zeta_i)$  are still holomorphic coordinates on  $X_{0,t''}$ . By the theory of Ahlfors and Bers [3] and Wolpert [7] we can assume that there are constants  $\delta, c > 0$  such that when  $|t''| < \delta$ ,  $\eta_i$  and  $\zeta_i$  are holomorphic coordinates on  $X_{0,t''}$  with  $0 < |\eta_i| < c$  and  $0 < |\zeta_i| < c$ . Now we assume  $t' = (t_1, \dots, t_m)$  has small norm. We do the plumbing construction on  $X_{0,t''}$  to obtain  $X_t = X_{t',t''}$ . For each  $i = 1, \dots, m$ , we remove the discs  $\{0 < |\eta_i| \leq \frac{|t_i|}{c}\}$  and  $\{0 < |\zeta_i| \leq \frac{|t_i|}{c}\}$  from  $X_{0,t''}$  and identify  $\{\frac{|t_i|}{c} < |\eta_i| < c\}$  with  $\{\frac{|t_i|}{c} < |\zeta_i| < c\}$  by the rule  $\eta_i \zeta_i = t_i$ . This defines the surface  $X_t$ . The tuple  $t = (t', t'') = (t_1, \dots, t_m, t_{m+1}, \dots, t_n)$  are the local pinching coordinates for the manifold cover of  $\overline{\mathcal{M}}_g$ . We call the coordinates  $\eta_i$  (or  $\zeta_i$ ) the plumbing coordinates on  $X_{t,s}$  and the collar  $\{\frac{|t_i|}{c} < |\eta_i| < c\}$  the plumbing collar.

**Remark 2.1.** From the estimate of Wolpert [14], [7] on the length of short geodesic, we have

$$u_i = \frac{l_i}{2\pi} \sim -\frac{\pi}{\log|t_i|}.$$

In [2] we first proved the equivalence of canonical metrics on  $\mathcal{M}_g$ :

**Theorem 2.1.** *All the canonical metrics on the moduli space  $\mathcal{M}_g$ : the Teichmüller-Kobayashi metric, the Carathéodory metric, the induced Bergman metric, the asymptotic Poincaré metric, the McMullen metric, the Ricci metric, the perturbed Ricci metric and the Kähler-Einstein metric are equivalent.*

The new metrics we defined have nice curvature properties which can be used to control the Kähler-Einstein metric. In [2] and [3] we proved

**Theorem 2.2.** *The Ricci and perturbed Ricci metrics are complete Kähler metrics with Poincaré growth. These metrics and the Kähler-Einstein metric have bounded geometry on the Teichmüller space  $\mathcal{T}_g$ . Furthermore, all the covariant derivatives of the curvature of the Kähler-Einstein metric are bounded. The Ricci and holomorphic sectional curvatures of the perturbed Ricci metric are bounded from above and below by negative constants.*

We also derived in [2] and [3] the precise asymptotic of the Weil-Petersson, Ricci and perturbed Ricci metrics and their curvature. This is one of the key components in the proof of the goodness of these metrics. For the Weil-Petersson and Ricci metrics we have

**Theorem 2.3.** *Let  $(t, s) = (t_1, \dots, t_m, s_{m+1}, \dots, s_n)$  be the pinching coordinates near a codimension  $m$  boundary point in  $\overline{\mathcal{M}}_g$ . Let  $h$  and  $\tau$  be the Weil-Petersson and Ricci metrics respectively. Then the Weil-Petersson metric has the asymptotic:*

- (1)  $h^{i\bar{i}} = 2u_i^{-3}|t_i|^2(1 + O(u_0))$  and  $h_{i\bar{i}} = \frac{1}{2}\frac{u_i^3}{|t_i|^2}(1 + O(u_0))$  for  $1 \leq i \leq m$ ;
- (2)  $h^{i\bar{j}} = O(|t_i t_j|)$  and  $h_{i\bar{j}} = O(\frac{u_i^3 u_j^3}{|t_i t_j|})$ , if  $1 \leq i, j \leq m$  and  $i \neq j$ ;
- (3)  $h^{i\bar{j}} = O(1)$  and  $h_{i\bar{j}} = O(1)$ , if  $m+1 \leq i, j \leq n$ ;
- (4)  $h^{i\bar{j}} = O(|t_i|)$  and  $h_{i\bar{j}} = O(\frac{u_i^3}{|t_i|})$  if  $i \leq m < j$ ;
- (5)  $h^{i\bar{j}} = O(|t_j|)$  and  $h_{i\bar{j}} = O(\frac{u_j^3}{|t_j|})$  if  $j \leq m < i$

where  $u_0 = \sum_{j=1}^m u_j + \sum_{j=m+1}^n |s_j|$ . The Ricci metric has the asymptotic:

- (1)  $\tau_{i\bar{i}} = \frac{3}{4\pi^2}\frac{u_i^2}{|t_i|^2}(1 + O(u_0))$  and  $\tau^{i\bar{i}} = \frac{4\pi^2}{3}\frac{|t_i|^2}{u_i^2}(1 + O(u_0))$ , if  $i \leq m$ ;
- (2)  $\tau_{i\bar{j}} = O(\frac{u_i^2 u_j^2}{|t_i t_j|}(u_i + u_j))$  and  $\tau^{i\bar{j}} = O(|t_i t_j|)$ , if  $i, j \leq m$  and  $i \neq j$ ;
- (3)  $\tau_{i\bar{j}} = O(\frac{u_i^2}{|t_i|})$  and  $\tau^{i\bar{j}} = O(|t_i|)$ , if  $i \leq m$  and  $j \geq m+1$ ;
- (4)  $\tau_{i\bar{j}} = O(1)$ , if  $i, j \geq m+1$ .

The holomorphic sectional curvature of the Ricci metric has the asymptotic:

- (1)  $\tilde{R}_{i\bar{i}i\bar{i}} = -\frac{3u_i^4}{8\pi^4|t_i|^4}(1 + O(u_0))$  if  $i \leq m$ ;
- (2)  $\tilde{R}_{i\bar{i}i\bar{i}} = O(1)$  if  $i > m$ .

We also have a weak curvature estimate of the Ricci metric. Let

$$\Lambda_i = \begin{cases} \frac{u_i}{|t_i|} & \text{if } i \leq m \\ 1 & \text{if } i > m. \end{cases}$$

Then

- (1)  $\tilde{R}_{i\bar{j}k\bar{l}} = O(1)$  if  $i, j, k, l > m$ ;
- (2)  $\tilde{R}_{i\bar{j}k\bar{l}} = O(\Lambda_i \Lambda_j \Lambda_k \Lambda_l) O(u_0)$  if at least one of these indices  $i, j, k, l$  is less than or equal to  $m$  and they are not all equal to each other.

The asymptotic of the perturbed Ricci metric and its curvature can be found in [2] and [3]. Also, precise estimates of the full curvature tensor of the Weil-Petersson, Ricci and perturbed

Ricci metrics, which will be used in the proof of their goodness, can be found in [5] and [6].

Stronger estimates of the asymptotics of these metrics lead to the Mumford's goodness condition of a singular Hermitian metrics on vector bundles over quasi-projective manifolds. We recall the definition and basic properties of good metrics from [1].

Let  $X$  be a quasi-projective variety of  $\dim_{\mathbb{C}} X = k$  obtained by removing a divisor  $D$  of normal crossings from a closed smooth projective variety  $\overline{X}$ . Let  $\overline{E}$  be a holomorphic vector bundle of rank  $n$  over  $\overline{X}$  and  $E = \overline{E}|_X$ . Let  $h$  be a Hermitian metric on  $E$  which may be singular near  $D$ .

We cover a neighborhood of  $D \subset \overline{X}$  by finitely many polydiscs

$$\{U_\alpha = (\Delta^k, (z_1, \dots, z_k))\}_{\alpha \in A}$$

such that  $V_\alpha = U_\alpha \setminus D = (\Delta^*)^m \times \Delta^{k-m}$ . Namely,  $U_\alpha \cap D = \{z_1 \cdots z_m = 0\}$ . We let  $U = \bigcup_{\alpha \in A} U_\alpha$  and  $V = \bigcup_{\alpha \in A} V_\alpha$ . On each  $V_\alpha$  we have the local Poincaré metric

$$\omega_{P,\alpha} = \frac{\sqrt{-1}}{2} \left( \sum_{i=1}^m \frac{1}{2|z_i|^2 (\log |z_i|)^2} dz_i \wedge d\bar{z}_i + \sum_{i=m+1}^k dz_i \wedge d\bar{z}_i \right).$$

**Definition 2.5.** Let  $\eta$  be a smooth local  $p$ -form defined on  $V_\alpha$ .

- We say  $\eta$  has Poincaré growth if there is a constant  $C_\alpha > 0$  depending on  $\eta$  such that

$$|\eta(t_1, \dots, t_p)|^2 \leq C_\alpha \prod_{i=1}^p \|t_i\|_{\omega_{P,\alpha}}^2$$

for any point  $z \in V_\alpha$  and  $t_1, \dots, t_p \in T_z X$ .

- We say  $\eta$  is good if both  $\eta$  and  $d\eta$  have Poincaré growth.

**Definition 2.6.** An Hermitian metric  $h$  on  $E$  is good if for all  $z \in V$ , assuming  $z \in V_\alpha$ , and for all basis  $(e_1, \dots, e_n)$  of  $\overline{E}$  over  $U_\alpha$ , if we let  $h_{i\bar{j}} = h(e_i, e_j)$ , then

- $|h_{i\bar{j}}|, (\det h)^{-1} \leq C (\sum_{i=1}^m \log |z_i|)^{2n}$  for some  $C > 0$  and  $n \geq 1$ .
- The local 1-forms  $(\partial h \cdot h^{-1})_{\alpha\gamma}$  are good on  $V_\alpha$ . Namely the local connection and curvature forms of  $h$  have Poincaré growth.

It is easy to see the following basic properties of good metrics:

- The definition of Poincaré growth is independent of the choice of  $U_\alpha$  or local coordinates on it.
- A form  $\eta \in A^p(X)$  with Poincaré growth defines a  $p$ -current  $[\eta]$  on  $\overline{X}$ . In fact we have

$$\int_X |\eta \wedge \xi| < \infty$$

for any  $\xi \in A^{k-p}(\overline{X})$ .

- If both  $\eta \in A^p(X)$  and  $\xi \in A^q(X)$  have Poincaré growth, then  $\eta \wedge \xi$  has Poincaré growth.
- For a good form  $\eta \in A^p(X)$ , we have  $d[\eta] = [d\eta]$ .

The importance of a good metric on  $E$  is that we can compute the Chern classes of  $\overline{E}$  via the Chern forms of  $h$  as currents. Namely, with the growth assumptions on the metric and its derivatives, we can integrate by part, so Chern-Weil theory still holds. In [1] Mumford has proved:



**Theorem 2.4.** *Given an Hermitian metric  $h$  on  $E$ , there is at most one extension  $\overline{E}$  of  $E$  to  $\overline{X}$  such that  $h$  is good.*

**Theorem 2.5.** *If  $h$  is a good metric on  $E$ , the Chern forms  $c_i(E, h)$  are good forms. Furthermore, as currents, they represent the corresponding Chern classes  $c_i(\overline{E}) \in H^{2i}(\overline{X}, \mathbb{C})$ .*

In the following sections, we will discuss the goodness of the above metrics and their applications.

### 3 Goodness of Canonical Metrics

In this section we describe the main ideas of proving the goodness of the Weil-Petersson, Ricci and perturbed Ricci metrics. From Theorem 2.3, it is very natural to consider the metrics induced by these metrics and the Kähler-Einstein metric on the logarithmic extension  $\overline{E} = T_{\overline{\mathcal{M}}_g}^*(\log D)$  of the cotangent bundle  $T_{\mathcal{M}_g}^*$  to the DM moduli space  $\overline{\mathcal{M}}_g$ .

We first give a general discussion of the goodness condition of the metric on  $\overline{E}$  induced by a Kähler metric  $g$  on  $\mathcal{M}_g$ . Let  $D = \overline{\mathcal{M}}_g \setminus \mathcal{M}_g$  be the compactification divisor and let  $p \in D$  be a codimension  $m$  boundary point in  $\overline{\mathcal{M}}_g$  with the corresponding stable nodal surface  $X_{0,0}$ . Let  $n = 3g - 3$  be the dimension of  $\mathcal{M}_g$ . Let  $(t_1, \dots, t_n)$  be the pinching coordinates near  $p$  where  $(t_1, \dots, t_m)$  corresponding to the degeneration directions.

For any Kähler metric  $g$  on  $\mathcal{M}_g$ , let  $g^*$  be the induced metric on  $\overline{E}$ . We know that

$$\left( \frac{dt_1}{t_1}, \dots, \frac{dt_m}{t_m}, dt_{m+1}, \dots, dt_n \right) \quad (3.1)$$

is a local holomorphic frame of  $\overline{E}$ . Under this frame, the metric  $g^*$  and its inverse are given by

$$g_{i\bar{j}}^* = \begin{cases} \frac{1}{t_i \bar{t}_j} g^{i\bar{j}} & i, j \leq m \\ \frac{1}{t_i} g^{i\bar{j}} & i \leq m < j \\ \frac{1}{\bar{t}_j} g^{i\bar{j}} & j \leq m < i \\ g^{i\bar{j}} & i, j > m \end{cases} \quad (3.2)$$

and

$$(g^*)^{i\bar{j}} = \begin{cases} t_i \bar{t}_j g_{i\bar{j}} & i, j \leq m \\ t_i g_{i\bar{j}} & i \leq m < j \\ \bar{t}_j g_{i\bar{j}} & j \leq m < i \\ g_{i\bar{j}} & i, j > m \end{cases} \quad (3.3)$$

Now we define two quantities. Let

$$D_i^k = \begin{cases} \frac{t_k}{t_i} & i, k \leq m \\ t_k & k \leq m < i \\ \frac{1}{t_i} & i \leq m < k \\ 1 & i, k > m \end{cases} \quad (3.4)$$

Let  $v_i = -\frac{\pi}{\log |t_i|}$  for  $i \leq m$  and let  $u_i = \frac{l_i}{2\pi}$  where  $l_i$  is the length of the geodesic loop on the  $i$ -th collar of  $X = X_t$ . We have that

$$u_i = v_i(1 + O(v_i)).$$

Now we let

$$\Lambda_i = \begin{cases} \frac{u_i}{|t_i|} & i \leq m \\ 1 & i > m \end{cases}. \quad (3.5)$$

Let  $A = (A_i^k)$  be the connection form of  $g^*$  where  $i$  is the row index and  $k$  is the column index. We have

$$A_i^k = \sum_p \left( \sum_j (\partial_p g_{i\bar{j}}^*) (g^*)^{k\bar{j}} \right) dt_p. \quad (3.6)$$

By (3.2), (3.3) and (3.4), we have

$$A_i^k = - \sum_p \left( D_i^k \sum_j (\partial_p g_{k\bar{j}}) g^{i\bar{j}} \right) dt_p \quad (3.7)$$

if  $i \neq k$  or  $i = k > m$ . We also have

$$A_i^i = - \sum_{p \neq i} \left( \sum_j (\partial_p g_{i\bar{j}}) g^{i\bar{j}} \right) dt_p - \left( \frac{1}{t_i} + \sum_j (\partial_i g_{i\bar{j}}) g^{i\bar{j}} \right) dt_i \quad (3.8)$$

if  $i = k \leq m$ .

To prove the goodness of  $g^*$ , the first order estimates are reduced to

$$\left| D_i^k \sum_j (\partial_p g_{k\bar{j}}) g^{i\bar{j}} \right| = O(\Lambda_p) \quad (3.9)$$

if  $i \neq k$  or  $i = k > m$  or  $p \neq i = k \leq m$  and

$$\left| \frac{1}{t_i} + \sum_j (\partial_i g_{i\bar{j}}) g^{i\bar{j}} \right| = O(\Lambda_i) \quad (3.10)$$

if  $i = k = p \leq m$ .

For the estimates on the  $g^*$  itself, we need to show that

$$|g_{i\bar{j}}^*|, (\det g^*)^{-1} \leq C \left( \sum_{i=1}^m \log |t_i| \right)^{2n}. \quad (3.11)$$

By (3.2) we have  $(\det g^*)^{-1} = |t_1 \cdots t_m|^2 (\det g)^{-1}$ , inequality (3.11) is equivalent to

$$|g_{i\bar{j}}^*|, |t_1 \cdots t_m|^2 (\det g)^{-1} \leq C \left( \sum_{i=1}^m \log |t_i| \right)^{2n}. \quad (3.12)$$

The second order estimates are reduced to show that  $dA_i^k$  has Poincaré growth for any choice of  $i, k$ . Since

$$dA = \partial A + \bar{\partial} A = \bar{\partial} A - A \wedge A,$$

if each entry of  $A$  has Poincaré growth, then each entry of  $A \wedge A$  has Poincaré growth. Thus we need to show that each entry of  $\bar{\partial}A$  has Poincaré growth.

By (3.7) and (3.8), since  $D_i^i = 1$ , we have

$$\bar{\partial}A_i^k = D_i^k \partial_{\bar{q}} \left( \sum_j (\partial_p g_{k\bar{j}}) g^{i\bar{j}} \right) dt_p \wedge d\bar{t}_q = -D_i^k g^{i\bar{j}} R_{k\bar{j}p\bar{q}} dt_p \wedge d\bar{t}_q \quad (3.13)$$

where  $R_{k\bar{j}p\bar{q}}$  is the curvature of  $g$ . Thus we need to show that

$$|D_i^k g^{i\bar{j}} R_{k\bar{j}p\bar{q}}| = O(\Lambda_p \Lambda_q). \quad (3.14)$$

By collecting the above argument, we have

**Lemma 3.1.** *The metric  $g^*$  on  $T_{\mathcal{M}_g}^*(\log D)$  induced by a Kähler metric  $g$  on  $\mathcal{M}_g$  is good if and only if the estimates (3.9), (3.10), (3.12) and (3.14) hold.*

Now We are ready to discuss the goodness of canonical metrics. The first theorem is

**Theorem 3.1.** *The metric  $h^*$  on the logarithm cotangent bundle  $\bar{E}$  over the DM moduli space induced by the WP metric is good in the sense of Mumford. Thus the Chern forms of  $h^*$ , as currents, are equal to the Chern classes of  $\bar{E}$ .*

We now sketch the proof of this theorem in three steps: the zero-th order, first order and second order estimates. The details are in [5]. In the following, we take the metric  $g_{i\bar{j}}$  to be the Weil-Petersson metric  $h_{i\bar{j}}$ . We use the same notation as in our paper [2].

We first consider the zero-th order estimate. This follows directly from Theorem 2.3.

**Lemma 3.2.** *The inequality (3.12) hold, for the Weil-Petersson metric  $h$ .*

*Proof.* By Theorem 2.3, (3.2) and (3.3), we have

$$h_{i\bar{j}}^* = \begin{cases} 2u_i^{-3}(1 + O(u_0)) & i = j \leq m \\ O(1) & \text{otherwise} \end{cases}$$

and

$$|t_1 \cdots t_m|^2 (\det h) \leq C \left( \prod_{i=1}^m u_i \right)^3.$$

It is easy to see that (3.12) holds.

Now we prove the first order estimates. In order to compute the connection of the induced metric  $h^*$ , it is easier to prove the formula (3.6) directly with the metric  $h$ . We use the estimate of Masur [15] and refine the estimates of Schumacher [4] and Trapani [16].

By the work of [2] we know that

$$h^{i\bar{j}} = \int_X \frac{\varphi_i \bar{\varphi}_j}{\lambda^2} dv \quad (3.15)$$

where  $\varphi_i$  is the holomorphic quadratic differential corresponding to  $dt_i$  and  $\lambda$  is the KE metric on  $X$ . In order to compute the connection forms of the WP metric, we need to estimate the derivatives of each  $\varphi_i$  and  $\lambda$ .

Unlike the approach in [2], here we take plumbing coordinate and plumbing collar rather than rs-coordinate and genuine collar because we need the trivialization of the collars. It is

easier to compute the derivative of the hyperbolic metric by using the plumbing coordinate on the degeneration collars.

We first change coordinate on the collars. Let  $(t_1, \dots, t_n)$  be the pinching coordinates near a codimension  $m$  boundary point  $p$  in the DM moduli. Let  $z_i$  and  $w_i$  be the plumbing coordinates on the  $i$ -th collar of  $X_t$  with  $i \leq m$ . Let  $r_i = |z_i|$ ,  $\theta_i = \arg z_i$ ,  $\tilde{r}_i = |w_i|$  and  $\tilde{\theta}_i = \arg w_i$ . We know that  $z_i w_i = t_i$ . Let  $\Omega_c^i$  be the  $i$ -th plumbing collar of size  $c$  with a fixed  $0 < c < 1$ . Namely,

$$\Omega_c^i = \{z_i \mid c^{-1}|t_i| \leq r_i \leq c\} = \{w_i \mid c^{-1}|t_i| \leq \tilde{r}_i \leq c\}.$$

We denote by  $\Omega_c$  the union of all collars:  $\Omega_c = \bigcup_{i=1}^m \Omega_c^i$ . We also define the half collars  $\Omega_c^{i+}$  and  $\Omega_c^{i-}$  by

$$\Omega_c^{i+} = \{z_i \mid |t_i|^{\frac{1}{2}} \leq r_i \leq c\}$$

and

$$\Omega_c^{i-} = \{z_i \mid c^{-1}|t_i| \leq r_i \leq |t_i|^{\frac{1}{2}}\} = \{w_i \mid |t_i|^{\frac{1}{2}} \leq \tilde{r}_i \leq c\}.$$

To compute the derivative of  $\varphi$ , by our works in [2] and the work of Masur [15], we have the expansion of  $\varphi_i$  on the plumbing collars. Let  $\Delta_\delta^n$  be the closed polydisc in  $\mathbb{C}^n$  such that the radius of each disk is  $\delta > 0$ . We assume the pinching coordinates  $t = (t_1, \dots, t_n)$  is defined for  $t \in \Delta_\delta^n$ . By shrinking  $\delta$  we have

**Lemma 3.3.** *Let  $k \leq m$  and let  $z_k$  and  $w_k$  be the plumbing coordinates on the  $k$ -th collar  $\Omega_{c_0}^k$  with  $c < c_0 < 1$  fixed. Then on  $\Omega_{c_0}^k$  we have*

- (1)  $\varphi_i = -\frac{t_i}{\pi} \frac{1}{z_k} (p_i^k(z_k) + q_i^k(z_k))$  if  $i \leq m$  and  $i \neq k$ ;
- (2)  $\varphi_i = \frac{1}{z_k} (p_i^k(z_k) + q_i^k(z_k))$  if  $i > m$ ;
- (3)  $\varphi_k = -\frac{t_k}{\pi} \frac{1}{z_k} (1 + p_k^k(z_k) + q_k^k(z_k))$ .

There is a constant  $M > 0$  such that in the above formulae, the functions  $p_i^k, q_i^k$  satisfy

- (1)  $p_i^k = \sum_{s=1}^{\infty} a_{is}^k(t) z_k^s$  such that each  $a_{is}^k(t)$  is a holomorphic function of the multi-variable  $t$  and  $\sum_{s=1}^{\infty} |a_{is}^k(t)| c_0^s \leq M$  for  $t \in \Delta_\delta^n$ ;
- (2)  $q_i^k = \sum_{s \leq -1} a_{is}^k(t) t_k^{-s} z_k^s$  such that  $a_{is}^k(t)$  is holomorphic in  $t$  and  $\sum_{s \leq -1} |a_{is}^k(t)| c_0^{-s} \leq M$  for  $t \in \Delta_\delta^n$ .

There are similar expansions by using the  $w_k$  coordinates. Furthermore, on  $X \setminus \Omega_c$  we have

$$\|\varphi_i\| = \begin{cases} O(|t_i|) & i \leq m \\ O(1) & i > m \end{cases}.$$

For the proof of this lemma, please see [15]. We also have the estimates of the derivatives of  $p_i^k$  and  $q_i^k$ :

**Lemma 3.4.** *Let  $0 < c < c_0$  be a fixed constant. On the collar  $\Omega_c^k$  we have*

- (1)  $\frac{\partial p_i^k}{\partial t_j} = \sum_{s=1}^{\infty} \frac{\partial a_{is}^k(t)}{\partial t_j} z_k^s$  such that  $\sum_{s=1}^{\infty} \left| \frac{\partial a_{is}^k(t)}{\partial t_j} \right| c^s \leq M_1$  for  $t \in \Delta_{\frac{\delta}{2}}^n$ ;
- (2)  $\frac{\partial q_i^k}{\partial t_j} = \sum_{s \leq -1} \frac{\partial a_{is}^k(t)}{\partial t_j} t_k^{-s} z_k^s$  such that  $\sum_{s \leq -1} \left| \frac{\partial a_{is}^k(t)}{\partial t_j} \right| c^{-s} \leq M_1$  for  $t \in \Delta_{\frac{\delta}{2}}^n$  and  $j \neq k$ ;
- (3)  $\frac{\partial q_i^k}{\partial t_k} = \frac{1}{t_k} \sum_{s \leq -1} b_{is}^k(t) t_k^{-s} z_k^s$  where  $b_{is}^k(t) = t_k \frac{\partial a_{is}^k(t)}{\partial t_k} - s a_{is}^k(t)$  and  $\sum_{s \leq -1} |b_{is}^k(t)| c^{-s} \leq M_1$ .

Here  $M_1$  is a constant depending on  $M, c, c_0, \delta, n$ .

By combining the above two lemmas, we can get desired estimates of the derivatives of each quadratic differential  $\varphi_i$ .

We then estimate the KE metric  $\lambda$  and its derivatives on each Riemann surface. The following estimate of  $\lambda$  is due to Masur [15]. The following lemma, although is not sharp, will be enough for our purpose.

**Lemma 3.5.** *For each  $1 \leq i \leq m$ , there is a constant  $\alpha > 0$  such that, on  $\Omega_c^{i+}$ , we have*

$$\frac{1}{\alpha} \frac{1}{r_i^2 (\log r_i)^2} \leq \lambda \leq \alpha \frac{1}{r_i^2 (\log r_i)^2}$$

and on  $\Omega_c^{i-}$ , we have

$$\frac{1}{\alpha} \frac{1}{\tilde{r}_i^2 (\log \tilde{r}_i)^2} \leq \lambda \leq \alpha \frac{1}{\tilde{r}_i^2 (\log \tilde{r}_i)^2}.$$

The estimate the derivative of  $\lambda$  is more subtle. We have

**Lemma 3.6.** *Let  $\lambda$  be the KE metric on the Riemann surface  $X = X_t$ . On each collar  $\Omega_c^k$ ,  $\lambda$  has a unique representation in term of the plumbing coordinate  $z_k$ . Then*

$$\left| \frac{\partial}{\partial t_i} (\log \lambda |_{\Omega_c}) \right| = O(\Lambda_i).$$

Some ideas of the proof of this lemma was drawn from the work of Schumacher in [4]. We briefly describe the proof here by using the compound graft metric constructed by Wolpert in [7].

Let  $\mathfrak{X}$  be the total space and let  $\pi : \mathfrak{X} \rightarrow \mathcal{M}_g$  be the projection. In [2], we established the curvature formulae of the WP metric and the Ricci metric by using the harmonic lift which directly gives the harmonic representatives of the Kodaira-Spencer classes. In this case, we need to use a different lift. For each  $i$ , let  $v_i$  be the harmonic lift of  $\frac{\partial}{\partial t_i}$ . Let  $\tilde{v}_i$  be a lift of  $\frac{\partial}{\partial t_i}$  such that  $\tilde{v}_i |_{\pi^{-1}(\Omega_c)} = \frac{\partial}{\partial t_i}$ . This can be done since we have a trivialization of  $\pi^{-1}(\Omega_c)$  by using the plumbing coordinates on the collars.  $\tilde{v}_i$  can be obtained by gluing fiberwisely an appropriately chosen lift of  $\frac{\partial}{\partial t_i}$  on  $X \setminus \Omega_c$  and the vector field  $\frac{\partial}{\partial t_i}$  on  $\Omega_c$  using a graft function.

Now let  $\tilde{\lambda}$  be the compound graft metric on  $X$ . A direct computation shows that

$$\left| \frac{L_{\tilde{v}_i} \tilde{\lambda}}{\tilde{\lambda}} - \frac{L_{\tilde{v}_i} \lambda}{\lambda} \right| = O(\Lambda_i)$$

and

$$\left| \frac{L_{\tilde{v}_i} \tilde{\lambda}}{\tilde{\lambda}} \right| = O(\Lambda_i).$$

The above two formulae imply

$$\left| \frac{L_{\tilde{v}_i} \lambda}{\lambda} \right| = O(\Lambda_i)$$

which is the conclusion since  $\tilde{v}_i |_{\pi^{-1}(\Omega_c)} = \frac{\partial}{\partial t_i}$ .

The estimates on  $\varphi_i$ ,  $\lambda$  and their derivatives give the estimates on the derivatives of the dual metric of the WP metric:

**Lemma 3.7.** *Let  $t = (t_1, \dots, t_n)$  be the local pinching coordinates with  $t \in \Delta_\delta^n$ . Assume  $\delta$  is small enough. If  $p > m$ , then*

$$\left| \partial_p h^{i\bar{j}} \right| = \begin{cases} O(|t_i|^2 u_i^{-3}) & i = j \leq m \\ O(|t_i t_j|) & i, j \leq m, i \neq j \\ O(|t_i|) & i \leq m < j \\ O(|t_j|) & j \leq m < i \\ O(1) & i, j > m \end{cases}. \quad (3.16)$$

If  $p \leq m$  and  $p \neq i$ , then

$$\left| \partial_p h^{i\bar{j}} \right| = \begin{cases} O(|t_i|^2 u_i^{-3} \Lambda_p) & i = j \leq m \\ O(|t_i t_j| \Lambda_p) & i, j \leq m, i \neq j \\ O(|t_i| \Lambda_p) & i \leq m < j \\ O(|t_j| \Lambda_p) & j \leq m < i \\ O(\Lambda_p) & i, j > m \end{cases}. \quad (3.17)$$

If  $p = i \leq m$  then

$$\left| \partial_i \left( \frac{1}{t_i} h^{i\bar{j}} \right) \right| = \begin{cases} O(u_i^{-2}) & i = j \leq m \\ O\left( \left| \frac{t_j}{t_i} \right| u_i^{-2} \right) & j \leq m, i \neq j \\ O(|t_i|^{-1} u_i^{-2}) & j > m \end{cases}. \quad (3.18)$$

We omit the proof of this lemma since it consists of very technique estimates. Please see [5] for details. Now the first order estimate follows from the above lemma, Theorem 2.3 and direct computations:

**Lemma 3.8.** *Let  $h^*$  be the metric on  $\bar{E}$  induced by the WP metric. Then*

$$\left| \sum_j \partial_p (h_{i\bar{j}}^*) \left( h^* \right)^{k\bar{j}} \right| = O(\Lambda_p)$$

for any  $i$  and  $k$ .

**Remark 3.1.** Let  $\Gamma_{ij}^k$  be the Christoffel symbol of the WP metric under the pinching coordinates. By (3.9) and (3.10), Lemma 3.8 is equivalent to

- (1)  $\left| \Gamma_{ii}^i + \frac{1}{t_i} \right| = O(\Lambda_i)$  if  $p = i = k \leq m$ ;
- (2)  $\left| D_i^k \Gamma_{kp}^i \right| = O(\Lambda_p)$  otherwise.

Finally we briefly discuss the second order estimate. We only give the main steps and omit the details. On one hand, since we need to estimate the curvature tensor of the Weil-Petersson metric, we can use the techniques in [2]. On the other hand, we need precise estimates which require us to repeat the work in [2] in an optimal way.

First of all, by Theorem 2.3, formula (3.14) is equivalent to the following two formulae:

$$\left| R_{\kappa\bar{j}p\bar{q}} \right| = \begin{cases} O(\Lambda_p \Lambda_q \frac{u_j^3}{|t_j|}) & j \leq m \\ O(\Lambda_p \Lambda_q) & j > m \end{cases} \quad (3.19)$$

if  $k > m$  and

$$|R_{k\bar{j}p\bar{q}}| = \begin{cases} O\left(\Lambda_p \Lambda_q \frac{u_j^3}{|t_j t_k|}\right) & j \leq m \\ O\left(\Lambda_p \Lambda_q \frac{1}{|t_k|}\right) & j > m \end{cases} \quad (3.20)$$

if  $k \leq m$ .

To check these two formulae, we use the curvature formula of the Weil-Petersson metric:

$$R_{k\bar{j}p\bar{q}} = - \int_X (e_{k\bar{j}} f_{p\bar{q}} + e_{k\bar{q}} f_{p\bar{j}}) dv. \quad (3.21)$$

In this case, we will use rs-coordinates on the genuine collars. We let  $X = X_t$  and let  $\Omega_c^i$  be the  $i$ -th genuine collar in  $X$  of size  $c$ . Let  $z_i$  be a rs-coordinate on  $\Omega_c^i$  with  $r_i = |z_i|$  and let  $\rho_i = e^{-\frac{2\pi^2}{t_i}}$ . By [2] we know that

$$\Omega_c^i = \{z_i \mid c^{-1}\rho_i \leq r_i \leq c\}.$$

We define the half collars

$$\Omega_c^{i+} = \{z_i \mid \rho_i^{\frac{1}{2}} \leq r_i \leq c\}$$

and

$$\Omega_c^{i-} = \{z_i \mid c^{-1}\rho_i \leq r_i \leq \rho_i^{\frac{1}{2}}\}.$$

For  $0 < c_1 < c$ , we define  $\Omega_c^j \setminus \Omega_{c_1}^j = \Gamma_1^j \cup \Gamma_2^j$  where

$$\Gamma_1^j = \{c_1 < r_j \leq c\}$$

and

$$\Gamma_2^j = \{c^{-1}\rho_j \leq r_j < c_1^{-1}\rho_j\}.$$

Let the functions  $\tilde{e}_{i\bar{j}}$  be defined as in [6]. The following technical lemma, as an optimal version of the proof of Corollary 4.2 of [2], is one of the key ingredient of the second order estimate.

**Lemma 3.9.** *Let  $j \leq m$ . Then  $\int_{\Omega_{c_1}^j} \tilde{e}_{j\bar{j}} f_{k\bar{k}} dv \geq 0$  and*

$$\int_{\Omega_{c_1}^j} \tilde{e}_{j\bar{j}} f_{k\bar{k}} dv = \begin{cases} O\left(\frac{u_j^4}{|t_j|^2}\right) & k > m \\ O\left(\frac{u_j^4}{|t_j|^2} \frac{u_k^6}{|t_k|^2}\right) & k \leq m, k \neq j \\ O\left(\frac{u_j^5}{|t_j|^4}\right) & k = m \end{cases} \quad (3.22)$$

By using this lemma, we have the second order estimate:

**Lemma 3.10.** *The formulae (3.19) and (3.20) hold.*

The proof of this part follows from Lemma 3.9 and a detailed case by case check. See [5]. Now we are ready to prove the main theorem 3.1.

*Proof.* By Lemma 3.2, 3.8 and 3.10 we already checked the goodness condition of the metric  $h^*$  by using the frame (3.1). If we choose another local holomorphic frame of  $\bar{E}$ , it is clear that

the change matrix is holomorphic and its determinant is non-zero and bounded. Thus the zero order, first order and second order estimates of using the new frame differ from the estimates of using the frame (3.1) by bounded terms. We finish the proof.

Based on the goodness of the Weil-Petersson metric, we can prove the goodness of the metrics on  $T_{\overline{\mathcal{M}}_g}^*(\log D)$  induced by the Ricci and perturbed Ricci metrics. These results here are more difficult than the goodness of the Weil-Petersson metric. Our second theorem of this section is

**Theorem 3.2.** *The metrics  $\tau^*$  and  $\tilde{\tau}^*$  on  $T_{\overline{\mathcal{M}}_g}^*(\log D)$  induced by the Ricci and perturbed Ricci metrics are good in the sense of Mumford.*

By Lemma 3.1, we need to show the estimates (3.9), (3.10), (3.12) and (3.14) hold when the metric  $g$  is the Ricci metric  $\tau$  or the perturbed Ricci metric  $\tilde{\tau}$ .

Here we only describe the proof of the estimates of the goodness of the Ricci metric  $\tau$ . Please see [5] for the perturbed Ricci metric.

In this case, the zero-th order estimate follows from Theorem 2.3 directly.

Now we consider the first order estimate. We let  $\tilde{\Gamma}_{ij}^k$  be the Christoffel symbol of the Ricci metric under the pinching coordinates. To show the first order estimate, we need to check that  $\tilde{\Gamma}_{ij}^k$  satisfy the inequalities (3.9) and (3.10). By Theorem 3.2 of [2] we have

$$\tilde{\Gamma}_{kp}^i = \tau^{i\bar{j}} h^{\alpha\bar{\beta}} \left( \sigma_1 \int_X \xi_p(e_{k\bar{j}}) e_{\alpha\bar{\beta}} dv \right) + \Gamma_{kp}^i \quad (3.23)$$

where  $\Gamma_{kp}^i$  is the Christoffel symbol of the WP metric. In order to prove the first order estimate for the Ricci metric, by Remark 3.1 and Lemma 3.8, we only need to estimate the tensor part. Thus the first order estimate is equivalent to

**Lemma 3.11.** *For any  $i, k$  and  $p$ , we have*

$$\left| \tau^{i\bar{j}} h^{\alpha\bar{\beta}} \left( \sigma_1 \int_X \xi_p(e_{k\bar{j}}) e_{\alpha\bar{\beta}} dv \right) \right| = O(\Lambda_p).$$

The central part of the proof of this lemma is following estimates which is the optimal version of the results in [2]:

**Lemma 3.12.**

$$\left| P \left( \tilde{e}_{i\bar{j}} \right) \right|_{L^1} = \begin{cases} O \left( \frac{u_i^3}{|t_i|^2} \right) & i = j \leq m \\ O \left( \frac{u_i^3 u_j^3}{|t_i t_j|} \right) & i, j \leq m, i \neq j \\ O \left( \frac{u_i^3}{|t_i|} \right) & i \leq m < j \\ O \left( \frac{u_j^3}{|t_j|} \right) & j \leq m < i \\ O(1) & i, j > m \end{cases}.$$

Similarly, the second order estimate follows from sharpening the estimates in [2]. Please see [5] for details.

It is well known that the line bundle  $K_{\overline{\mathcal{M}}_g} + [D]$ , which is the determinant bundle of  $T_{\overline{\mathcal{M}}_g}^*(\log D)$ , is positive. In [3] we proved



**Theorem 3.3.** *The logarithmic cotangent bundle  $\overline{E} = T_{\overline{\mathcal{M}}_g}^*(\log D)$  is stable with respect to its first Chern class.*

Since  $\overline{E}$  is stable, by the works of Donaldson-Uhlenbeck-Yau [17], [18] there is a Hermitian-Einstein metric on  $\overline{E}$  with respect to any metric in the class  $c_1(\overline{E})$  which gives an orbifold Chern number inequality. However, the metric induced by the Kähler-Einstein metric, if it is good, will give a stronger Chern number inequality. Thus a more interesting question is whether the metric on  $T_{\overline{\mathcal{M}}_g}^*(\log D)$  induced by the Kähler-Einstein metric on  $\mathcal{M}_g$  is good or not. The difficulty arises from that it is vary hard to control the off-diagonal terms in the local expression of the Kähler-Einstein metric with respect to the pinching coordinates. We believe that this difficulty can be solved by studying the Kähler-Ricci flow on  $\mathcal{M}_g$  with the Ricci metric as the initial metric and showing that the goodness is preserved under the flow.

#### 4 Dual Nakano Negativity of the Weil-Petersson metric

The Weil-Petersson metric has many negative curvature properties. Ahlfors showed that its Riemannian sectional curvature is negative. Later, Schumacher showed in [4] that the curvature of the WP metric is strongly negative in the sense of Siu. In this section, we prove that WP metric is dual Nakano negative from which we will derive Nakano-type vanishing theorems in next section.

We first recall the concept of dual Nakano negativity. Let  $(E^m, h)$  be a holomorphic vector bundle with a Hermitian metric over a complex manifold  $M^n$ . The curvature of  $E$  is given by

$$P_{i\bar{j}\alpha\bar{\beta}} = -\partial_\alpha\partial_{\bar{\beta}}h_{i\bar{j}} + h^{p\bar{q}}\partial_\alpha h_{i\bar{q}}\partial_{\bar{\beta}}h_{p\bar{j}}.$$

$(E, h)$  is Nakano semi-positive if the curvature  $P$  defines a semi-positive form on the bundle  $E \otimes T_M$ . Namely,

$$P_{i\bar{j}\alpha\bar{\beta}}C^{i\alpha}\overline{C^{j\beta}} \geq 0 \tag{4.1}$$

for all  $m \times n$  complex matrix  $C$ . The metric  $h$  is Nakano positive if (4.1) is a strict inequality whenever  $C \neq 0$ .  $E$  is dual Nakano (semi) negative if the dual bundle with the induced metric  $(E^*, h^*)$  is Nakano (semi) positive.

Our main result in this section is

**Theorem 4.1.** *Let  $\mathcal{M}_g$  be the moduli space of Riemann surfaces of genus  $g \geq 2$ . Then  $(T_{\mathcal{M}_g}, \omega_{WP})$  is semi-Nakano negative.*

To prove that the WP metric  $h$  on the tangent bundle of  $\mathcal{M}_g$  is dual Nakano negative, we only need to show that  $(T^*\mathcal{M}_g, h^*)$  is Nakano positive. Let  $R_{i\bar{j}k\bar{l}}$  be the curvature of  $\mathcal{M}_g$  and  $P_{i\bar{j}k\bar{l}}$  be the curvature of the cotangent bundle. We first have

$$P_{m\bar{n}k\bar{l}} = -h^{i\bar{m}}h^{m\bar{j}}R_{i\bar{j}k\bar{l}}.$$

Thus if we let  $a_{kj} = \sum_m h^{m\bar{j}}C^{mk}$ , we have

$$P_{m\bar{n}k\bar{l}}C^{mk}\overline{C^{nl}} = -\sum_{i,j,k,l} R_{i\bar{j}k\bar{l}}a_{kj}\overline{a_{li}} = -\sum_{i,j,k,l} R_{k\bar{j}i\bar{l}}a_{kj}\overline{a_{li}} = -\sum_{i,j,k,l} R_{i\bar{j}k\bar{l}}a_{ij}\overline{a_{lk}}.$$

Recall that at  $X \in \mathcal{M}_g$  we have

$$R_{i\bar{j}k\bar{l}} = - \int_X (e_{i\bar{j}} f_{k\bar{l}} + e_{i\bar{l}} f_{k\bar{j}}) dv.$$

By combining the above two formulae, to prove that the WP metric is dual Nakano negative is equivalent to show that

$$\int_X (e_{i\bar{j}} f_{k\bar{l}} + e_{i\bar{l}} f_{k\bar{j}}) a_{ij} \overline{a_{lk}} dv \geq 0 \quad (4.2)$$

and the left hand side of the above formula is strictly positive if  $A = [a_{ij}] \neq 0$ .

We now describe the proof with the assumption that the matrix  $[a_{ij}]$  is invertible. The general case can be found in [5] which follows from the same idea.

Recall that if we let  $\square = -\lambda^{-1} \partial_z \partial_{\bar{z}}$  be the Laplace operator with respect to the KE metric  $\lambda$  on  $X$  and let  $T = (\square + 1)^{-1}$ , then  $e_{i\bar{j}} = T(f_{i\bar{j}})$  where  $f_{i\bar{j}} = A_i \overline{A_j}$  and  $A_i$  is the harmonic representative of the Kodaira-Spencer class of  $\frac{\partial}{\partial t_i}$  where  $(t_1, \dots, t_n)$  are local coordinates on  $\mathcal{M}_g$  and  $z$  is the local coordinate on  $X_t$ .

Let  $B_j = \sum_{i=1}^n a_{ij} A_i$ . Then the inequality (4.2) is equivalent to

$$- \sum_{j,k} R(B_j, \overline{B_k}, A_k, \overline{A_j}) = \sum_{j,k} \int_X (T(B_j \overline{A_j}) A_k \overline{B_k} + T(B_j \overline{B_k}) A_k \overline{A_j}) dv \geq 0. \quad (4.3)$$

Since  $\{A_k\}$  is a basis of the space  $H^{0,1}(X, T_X)$  and the matrix  $[a_{ij}]$  is an arbitrary invertible matrix, we need to show that the inequality (4.3) holds for any two bases  $\{A_i\}$  and  $\{B_i\}$ . Of course we can choose one basis, say  $\{A_i\}$ , and let the other basis vary freely.

Now we prove the inequality (4.3). Let  $\mu = \sum_j B_j \overline{A_j} \in C^\infty(X)$ . Then the first term in (4.3) is

$$\begin{aligned} \sum_{j,k} \int_X T(B_j \overline{A_j}) A_k \overline{B_k} dv &= \int_X T(\mu) \overline{\mu} dv = \int_X T(\mu) (\square + 1) (T(\overline{\mu})) dv \\ &= \int_X (|T(\mu)|^2 + |\nabla(T(\mu))|^2) dv \geq 0 \end{aligned}$$

where  $|\nabla(T(\mu))|^2 = \lambda^{-1} \partial(T(\mu)) \overline{\partial(T(\mu))}$ .

To check the second term, we let  $G(z, w)$  be the Green's function of the operator  $T$ . Namely, for any function  $f \in C^\infty(X)$ , we have  $T(f) = \int_X G(z, w) f(w) dv(w)$ . Now we let

$$H(z, w) = \sum_j \overline{A_j}(z) B_j(w).$$

We know the second term of (4.3) is

$$\begin{aligned} \sum_{j,k} \int_X T(B_j \overline{B_k}) A_k \overline{A_j} dv &= \sum_{j,k} \int_{X \times X} G(z, w) B_j(w) \overline{B_k}(w) A_k(z) \overline{A_j}(z) dv(w) dv(z) \\ &= \int_{X \times X} G(z, w) H(z, w) \overline{H}(z, w) dv(w) dv(z) \geq 0 \end{aligned} \quad (4.4)$$

where the last inequality follows from the fact that the Green's function  $G$  is non-negative which is proved by Wolpert in [8], page 136.

Here we note that, although  $H(z, w)$  is not a global function, the quantity

$$H(z, w)\overline{H}(z, w) = (B_j(w)\overline{B}_k(w))(A_k(z)\overline{A}_j(z)) \in C^\infty(X \times X)$$

is a well-defined function. Also, it is clear that this function is non-negative and thus inequality (4.4) hold naturally.

**Remark 4.1.** It is possible to show that the singular metric on  $T_{\overline{\mathcal{M}}_g}(-\log D)$  induced by the Weil-Petersson metric is also dual Nakano negative. This follows from the dual Nakano negativity of the WP metric and simple linear algebra.

### 5 Applications: $L^2$ -Cohomology and Vanishing Theorems

The dual Nakano negativity of a Hermitian metric on a bundle over a compact manifold gives strong vanishing theorems by using Bochner techniques. However, in our case the base variety  $\mathcal{M}_g$  is only quasi-projective. Thus we can only describe vanishing theorems of the  $L^2$ -cohomology.

In [19], Saper showed that the  $L^2$ -cohomology of the moduli space equipped with the Weil-Petersson metric can be identified with the ordinary cohomology of the DM moduli space. Our situation is more subtle since the natural object to be considered in our case is the tangent bundle valued  $L^2$ -cohomology. Generalizing Saper's work, we proved in [5]

**Theorem 5.1.** *We have the following natural isomorphism:*

$$H_{(2)}^*((\mathcal{M}_g, \omega_\tau), (T_{\mathcal{M}_g}, \omega_{WP})) \cong H^*(\overline{\mathcal{M}}_g, T_{\overline{\mathcal{M}}_g}(-\log D)) \quad (5.1)$$

where  $\omega_\tau$  is the Ricci metric on  $\mathcal{M}_g$ .

We first explain this theorem. To define the  $L^2$ -cohomology of a vector bundle over a manifold, we need metrics on both the bundle and the manifold. Here we view  $T_{\mathcal{M}_g}$  purely as a bundle over the moduli space  $\mathcal{M}_g$ . We use the WP metric as the bundle metric and we put the Ricci metric on the base variety  $\mathcal{M}_g$ . We do this because of the technique difficulty that the WP metric is incomplete and thus there is trouble in defining the adjoint operator  $\overline{\partial}^*$ . After setting up the section spaces appropriately, the proof of this theorem is a direct application of the goodness of these metrics and can be found in [5].

Now we combine the above result with the dual Nakano negativity of the Weil-Petersson metric. In [5] we proved the following Nakano-type vanishing theorem

**Theorem 5.2.** *The  $L^2$ -cohomology group vanish:*

$$H_{(2)}^{0,q}((\mathcal{M}_g, \omega_\tau), (T_{\mathcal{M}_g}, \omega_{WP})) = 0 \quad (5.2)$$

unless  $q = 3g - 3$ .

The proof is similar to proof of Nakano vanishing theorem in the case when the base manifold is closed. It depends on the Kodaira-Nakano identity

$$\square_{\overline{\partial}} = \square_{\nabla} + \sqrt{-1}[\nabla^2, \Lambda].$$

We then apply the dual Nakano negativity of the WP metric to get the vanishing theorem by using the goodness to deal with integration by part to show that there is no boundary term.

We note here that, in the proof of Nakano vanishing theorem, we only need the negativity of the curvature of the bundle metric.

The above two theorems imply a result of Hacking [20]

**Corollary 5.1.** *The pair  $(\overline{\mathcal{M}}_g, D)$  is infinitesimally rigid.*

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