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d-dimensional unitarity cut method

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Abstract

We develop a unitarity method to compute one-loop amplitudes with massless propagators in $d = 4 - 2\epsilon$ dimensions. We compute double cuts of the loop amplitudes via a decomposition into a four-dimensional and a -2ϵ-dimensional integration. The four-dimensional integration is performed using spinor integration or other efficient techniques. The remaining integral in -2ϵ dimensions is cast in terms of bubble, triangle, box, and pentagon master integrals using dimensional shift identities. The method yields results valid for arbitrary values of ϵ.

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1. Introduction

In modern collider experiments complex events with multi-jets, vector bosons and jets, top quarks and jets, etc. are frequently produced. Their quantitative theoretical description requires cross-sections calculated at the one-loop level and even beyond. There exist mature techniques solving all conceptual problems which arise in one-loop computations. However, calculating one-loop multi-leg amplitudes with standard methods is tedious, due to the large number of Feynman diagrams and the algebraic complexity of tensor reduction. In recent years, new attempts are being made to replace or improve traditional approaches with more efficient and better automated methods. Significant progress can be expected with the advent of new powerful techniques.

Unitarity cuts of loop amplitudes have been introduced as an efficient tool to calculate QCD amplitudes [1]. A new four-dimensional unitarity method was developed recently [2,3], building on techniques inspired by twistor space geometry [4–11]. The phase-space integration is carried out explicitly in terms of spinors. The result is easily mapped to bubble, triangle, and box master integrals using analyticity properties. Many, mostly supersymmetric, amplitudes can be reconstructed fully with this technique [2,3,12] and other unitarity methods in four dimensions. However, the mapping to master integrals is in general incomplete, since rational contributions arising from multiplying $1/\epsilon$ poles of master integrals with $\mathcal{O}(\epsilon)$ coefficients are not accounted for.

New methods to compute the rational parts separately were introduced recently. They compute these terms by either developing [13–17] recursion relations for amplitudes [18,19], or by using specialized diagrammatic reductions [20–23]. As a result, for example, short analytic formulas are now available for all the one-loop six gluon QCD helicity amplitudes.

It was recognized long ago [24] that one can reconstruct the full amplitudes from unitarity cuts in $d = 4 - 2\epsilon$ dimensions. A complete method for one-loop calculations was developed in the pioneering work of Bern et al. [25–27], and it was recently re-examined in [28]. However, the calculation of general unitarity cuts remains formidable. While it is simpler than a direct Feynman graph evaluation, eventually, one resorts to traditional reduction methods to complete their computation.

In this Letter, we develop an efficient *d*-dimensional unitary cut method, reducing one-loop amplitudes to master inte-

grals for arbitrary values of the dimension parameter. We can read out the coefficient of the master integrals without fully carrying out the d-dimensional phase space integrals. Only a four-dimensional integration is explicitly required; we show how to perform this using spinor integration for *light-like* momenta [2,3,10,11]. A remaining integral, which gives rise to the ϵ-dependence of the cut-amplitude, is mapped to phase-space integrals in $4 + 2n - 2\epsilon$ dimensions, where n is a positive integer. With recursive dimensional shift identities, similar to the ones in loop integration, we reduce the cut-amplitude in terms of bubble, triangle, box and pentagon cut master integrals in $4 - 2\epsilon$ dimensions. The reduction is valid for an arbitrary number of dimensions. Expanding in ϵ, we can obtain both the (poly)logarithmic and rational part of the amplitude at $\mathcal{O}(\epsilon^0)$ and higher; these contributions are required in cross-sections beyond the next-to-leading order in the relevant coupling strength.

The core part of our method is the four-dimensional integration, where we have primarily used spinor integration. In [3,12] it was demonstrated that this method is very efficient and yields compact results for the cut-constructible part of multi-leg QCD amplitudes. In our d-dimensional unitarity case the four-dimensional integrand depends on an additional mass parameter. While the size of expressions is larger, spinor integration works efficiently by preserving gauge invariance at intermediate stages of the computation. The final results remain compact.

2. Reduction to master integrals

We consider one-loop amplitudes with massless internal propagators in the four-dimensional helicity (FDH) scheme; all external momenta are in four dimensions and the loop momentum in $d = 4 - 2\epsilon$. We shall reduce double cuts of the amplitude to master integrals, for arbitrary values of ϵ.

The basic quantity that we require is a generic double cut of the amplitude in $4 - 2\epsilon$ dimensions:

$$\mathcal{M} = \int d^{4-2\epsilon} p\, \delta(p^2)\delta\big((K-p)^2\big)\mathcal{A}_L(p)\mathcal{A}_R(p), \tag{1}$$

where $\mathcal{A}_{L,R}$ are tree amplitudes, and K is the sum of the momenta of the cut propagators. Since external momenta are in four dimensions, we can decompose the loop momentum as $p = \tilde{\ell} + \vec{\mu}$, where $\tilde{\ell}$ is 4-dimensional and $\vec{\mu}$ is (-2ϵ)-dimensional [25–27]. One can immediately perform the angular integrations for $\vec{\mu}$, yielding:

$$\mathcal{M} = \frac{\pi^{-\epsilon}}{\Gamma(-\epsilon)} \int d\mu^2 \left(\mu^2\right)^{-1-\epsilon}$$
$$\times \int d^4\tilde{\ell}\, \delta\big(\tilde{\ell}^2 - \mu^2\big)\delta\big((K-\tilde{\ell})^2 - \mu^2\big)$$
$$\times \mathcal{A}_L(\tilde{\ell} + \vec{\mu})\mathcal{A}_R(\tilde{\ell} + \vec{\mu}). \tag{2}$$

The unitarity cut integral of massless particles living in d dimensions is decomposed into a unitary cut integral of massive particles in four dimensions, and an integral over the mass parameter regularized with ϵ.

We now perform the integration over $\tilde{\ell}$. Given the virtues of spinor integration, it is desirable to employ it here. However, the

method is formulated for phase-space integrations of light-like particles and, at first sight, is not applicable to our case. We can find a generalization to the phase-space of massive particles, if we decompose the momentum $\tilde{\ell}$ into a linear combination of a light-like vector and the time-like cut-momentum K: $\tilde{\ell} = \ell + zK$, with $\ell^2 = 0$. The massive phase space integral turns into massless:

$$\int d^4\tilde{\ell}\, \delta(\tilde{\ell}^2 - \mu^2)\delta\big((\tilde{\ell} - K)^2 - \mu^2\big)$$
$$\rightarrow \int dz\,(1-2z)K^2\delta\big(z(1-z)K^2 - \mu^2\big)$$
$$\times \int d^4\ell\, \delta^+(\ell^2)\delta\big((1-2z)K^2 - 2K \cdot \ell\big). \tag{3}$$

The last line is the familiar phase space integration for two massless cut propagators; the only difference is the factor $(1-2z)$ appearing in the second delta function. The z-integral is trivially performed using the delta-function that is independent of ℓ. Thus we get $z = (1 - \sqrt{1-u})/2$, where $u = 4\mu^2/K^2 \in [0,1]$.

Following [3,5,10] we transform into spinor variables, so that $\ell^{a\dot{a}} = t\lambda^a \tilde{\lambda}^{\dot{a}}$. The phase-space measure, up to an overall normalization factor, becomes:

$$\int du\, u^{-1-\epsilon} \int \langle \lambda\, d\lambda\rangle[\lambda\, d\lambda],$$
$$\int_0^\infty t\, dt\, \delta\big(\sqrt{1-u}\, K^2 + t\langle\lambda|K|\lambda]\big). \tag{4}$$

The spinor integration can be carried out easily. The basic steps involve the application of Schouten identities in order to eliminate $\tilde{\lambda}^{\dot{a}}$ from the numerator of the integrand, and to locate holomorphic anomalies, reading out the result of the integration as a finite sum of residues. We refer the reader to [3,12] for a detailed description of the technique.

After spinor integration, we are left with a single integral over u,

$$\mathcal{M} = \int_0^1 du\, u^{-1-\epsilon} \sum_i f_i(u)\mathcal{L}_i(u), \tag{5}$$

where the coefficients $f_i(u)$ are rational functions of u. The functions $\mathcal{L}_i$ are combinations of logarithmic and square root functions with characteristic analyticity properties; they correspond to the analytic expressions of massive cut master integrals (bubbles, triangles, and boxes) in four dimensions.

We can express the cut amplitude $\mathcal{M}$ in terms of master integrals in $4 - 2\epsilon$ dimensions without explicitly performing the integration in Eq. (5). Many coefficients $f_i(u)$ are simple polynomials in u. All such terms are easily identified as one-loop master integrals in dimensions shifted by an even number $2n$. Schematically, bubble, triangle, and box master integrals emerge in the form:

$$\text{Bub}^{(n)} = \int_0^1 du\, u^{-1-\epsilon} u^n \sqrt{1-u}, \tag{6}$$

$$\text{Tri}^{(n)} = \int_0^1 du\, u^{-1-\epsilon} u^n \ln\left(\frac{Z + \sqrt{1-u}}{Z - \sqrt{1-u}}\right), \qquad (7)$$

$$\text{Box}^{(n)} = \int_0^1 du\, u^{-1-\epsilon} \frac{u^n}{\sqrt{B - Au}}$$

$$\times \ln\left(\frac{D - Cu - \sqrt{1-u}\sqrt{B-Au}}{D - Cu + \sqrt{1-u}\sqrt{B-Au}}\right), \qquad (8)$$

where Z^2, A, B, C, D are rational functions of kinematical invariants of the external momenta. Details of the exact expressions will be given in a forthcoming publication. While mapping to ϵ-dependent master integrals, a term of the form u^n is always absorbed into the measure factor $u^{-1-\epsilon}$, producing a dimensional shift.

After partial fractioning and identifying all $(4 - 2\epsilon + 2n)$-dimensional bubble, triangle, and box master integrals, a few coefficients f_i are yet not mapped to any master integral. These coefficients have a u-dependent monomial in the denominator, and multiply logarithms originating from box master integrals in four dimensions. They are related to the pentagon scalar integral, which can be expressed as a sum of box integrals in four dimensions plus some term in higher order of ϵ. However, it is a master in arbitrary dimensions. Upon integrating over u, the remaining terms combine to give rise to pentagon master integrals in $4 - 2\epsilon$ dimensions.

As a last step, we reduce the master integrals in $4 - 2\epsilon + 2n$ dimensions, to master integrals in $4 - 2\epsilon$ dimensions. We can derive compact dimensional shift identities for the phase-space master integrals from the representations in Eqs. (6)–(8) using integration by parts. These identities are equivalent to dimensional shift identities for loop integrals [36–38]. We will give their explicit analytic form and a simple derivation in a forthcoming publication. Here we just present the results.

$$\text{Bub}^{(n)} = F_{2\to2}^{(n)}\text{Bub}^{(0)},$$

$$\text{Tri}^{(n)}(Z) = F_{3\to3}^{(n)}(Z)\text{Tri}^{(0)}(Z) + F_{3\to2}^{(n)}(Z)\text{Bub}^{(0)},$$

$$\text{Box}^{(n)} = F_{4\to4}^{(n)}\text{Box}^{(0)} + \left\{F_{4\to3}^{(n)}(Z_1)\text{Tri}^{(0)}(Z_1)\right.$$

$$\left. + F_{4\to2}^{(n)}(Z_1)\text{Bub}^{(0)} + (Z_1 \leftrightarrow Z_2)\right\},$$

$$F_{2\to2}^{(n)} = \frac{(-\epsilon)_{3/2}}{(n-\epsilon)_{3/2}}, \qquad F_{3\to3}^{(n)} = \frac{-\epsilon}{n-\epsilon}(1 - Z^2)^n,$$

$$F_{4\to4}^{(n)} = \frac{(-\epsilon)_{1/2}}{(n-\epsilon)_{1/2}}\left(\frac{B}{A}\right)^n,$$

$$F_{3\to2}^{(n)} = \frac{(-\epsilon)_{3/2}}{n-\epsilon}\sum_{k=1}^n \frac{2Z(1 - Z^2)^{n-k}}{(k-\epsilon)_{1/2}},$$

$$F_{4\to j}^{(n)} = \frac{D + (Z^2 - 1)C}{(n-\epsilon)_{1/2}ZA}\sum_{k=1}^n\left(\frac{B}{A}\right)^{n-k}\frac{F_{3\to j}^{(k-1)}}{(k-1/2-\epsilon)_{1/2}}. \qquad (9)$$

Here $(x)_n = \Gamma(x+n)/\Gamma(x)$, and $j = 2, 3$. Z_1, Z_2 correspond to the two possible cut-triangles obtained by pinching the uncut propagators of the box.

In this Letter, we have limited our work to one-loop amplitudes with massless internal propagators. However, the method can be extended to the massive case. The spinor integration method is already adapted to massive phase-space. The only remaining issue is to find the reduction coefficient of master integrals with only one loop propagator; these vanish when a two-particle phase-space is considered. However, such terms are significantly constrained and often fully determined from the known ultraviolet and infrared behavior of one-loop amplitudes [25]. We will investigate this issue further in a future publication.

3. Alternatives to spinor integration

We have seen that our calculations can be divided into two steps: the four-dimensional-massive cut-integration and the dimensional shift. We have the flexibility of using alternative methods for the four-dimensional integration to compute the coefficients $f_i(u)$ in Eq. (5). Following a more traditional approach, we could apply the phase-space reduction methods of [29]. The integrals we consider here are free from both infrared and ultraviolet singularities, and no dimensional regulator is required. In precisely four dimensions, the reduction is much less tedious than in arbitrary dimensions. In many complicated cases it can be performed analytically, and in all cases of practical interest it can also be executed numerically [30–34]. This technique provides a valuable cross-check on our results with spinor integration.

Another appealing idea has appeared recently in the literature. Ossola, Papadopoulos and Pittau (OPP) introduced a purely algebraic procedure to compute master integral coefficients at the integrand level [35]. One can adapt the same technique for the four-dimensional phase-space integration over cut amplitudes. As an ingredient of the d-dimensional unitarity method, it should be a very efficient tool to analytically compute one-loop amplitudes in arbitrary dimensions.

OPP investigated the most general analytic form of one-loop amplitudes in four dimensions. The integrands of one-loop amplitudes are decomposed as:

$$A(\tilde{\ell}) = \sum_i\left(c_i + \sum_j S_{ij}(\tilde{\ell})b_{ij}\right)I_i(\tilde{\ell}), \qquad (10)$$

where I_i are products of propagators corresponding to scalar master integrals, and c_i, b_{ij} are constant coefficients. The universal terms $S_{ij}(\tilde{\ell})$ are "spurious" and yield a zero contribution to the amplitude after integration $\int d^4\tilde{\ell}\, S_{ij}(\tilde{\ell})I_i(\tilde{\ell}) = 0$; they are known explicitly for all master integrals [35].

With the analytic form of Eq. (10) at hand, the coefficients c_i, b_{ij} can be evaluated by computing the integrand algebraically at sufficiently many values of the loop momentum and forming a linear system of equations. The method is optimized by choosing values of the loop momenta that correspond to cuts of the loop amplitude, setting denominators in the master integrals to zero. In this way, the system is divided into closed subsystems for the integrals which survive any specific cut and can be solved easily.

In sum, we can apply any efficient method, for example the method of [35], to find the coefficients $f_i(u)$ in Eq. (5).[1] Note that, in this way, we can also compute the contributions to one-propagator master integrals which appear in amplitudes with massive propagators.

4. Summary

In this Letter, we have presented a new unitarity method for the reduction of one-loop amplitudes to master integrals in arbitrary dimensions. We have generalized the method of spinor integration via the holomorphic anomaly to massive phase-space integrals. The method consists of an explicit four-dimensional integration over the phase-space of double-cut amplitudes, and a remaining integration over a mass parameter.

As a cross-check of the four-dimensional integration, one may employ traditional, numerical and analytic phase-space reductions. Recently, an elegant proposal to compute the reduction coefficients of one-loop amplitudes has appeared in the literature [35]. This proposal may also be adopted within our method in order to perform the four-dimensional integration.

The final integration over a mass parameter is mapped directly to phase-space master integrals with shifted dimensions. A full reduction to master integrals in $4 - 2\epsilon$ is achieved with compact dimensional shift identities.

We anticipate our method to be useful for a wide spectrum of processes at colliders.

Acknowledgements

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