Matrioska: A Compiler for Multi-Key Homomorphic Signatures

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Abstract. Multi-Key Homomorphic Signatures (MK-HS) enable clients in a system to sign and upload messages to an untrusted server. At any later point in time, the server can perform a computation C on data provided by t different clients, and return the output y and a short signature $\sigma_{C,\mathsf{y}}$ vouching for the correctness of y as the output of the function C on the signed data. Interestingly, MK-HS enable verifiers to check the validity of the signature using solely the public keys of the signers whose messages were used in the computation. Moreover, the signatures $\sigma_{C,\mathsf{y}}$ are succinct, namely their size depends at most linearly in the number of clients, and only logarithmically in the total number of inputs of C. Existing MK-HS are constructed based either on standard assumptions over lattices (Fiore *et al.*, ASIACRYPT'16), or on non-falsifiable assumptions (SNARKs) (Lai *et al.*, ePrint'16). In this paper, we investigate connections between single-key and multi-key homomorphic signatures. We propose a generic compiler, called Matrioska, which turns any (sufficiently expressive) single-key homomorphic signature scheme into a multi-key scheme. Matrioska establishes a formal connection between these two primitives and is the first alternative to the only known construction under standard falsifiable assumptions. Our result relies on a novel technique that exploits the homomorphic property of a single-key HS scheme to compress an arbitrary number of signatures from t different users into only t signatures.

1 Introduction

Consider a scenario where a user Alice uploads a collection of data items $x_1, \ldots, x_\mathsf{n}$ to an untrusted server. Later on, the server executes a computation $\mathcal{P}$ on this data and sends the result $\mathsf{y} = \mathcal{P}(x_1, \ldots, x_\mathsf{n})$ to another user Bob.

How can Bob be sure that y *is the correct result obtained by running* $\mathcal{P}$ *on Alice's data?*

A trivial solution to this problem could be obtained by employing digital signatures: Alice could sign each data item x_i and send to the server the signatures $\sigma_1, \ldots, \sigma_\mathsf{n}$. Next, to convince Bob, a server can send along with y the original inputs with their signatures, and Bob should check that $\mathsf{y} = \mathcal{P}(x_1, \ldots, x_\mathsf{n})$ and that each σ_i is a valid signature for x_i. While this solution solves the above security concern, it has a clear efficiency drawback: it requires communication between the server and the verifier Bob that is *linear* in the input size of $\mathcal{P}$. This linear cost is not only undesirable but can be also unacceptable if Bob is not able to store the whole dataset.

Homomorphic Signatures. A solution to the above problem that achieves both security and efficiency can be obtained by using *homomorphic signatures* (HS). With this primitive, Alice can use her secret key to sign $x_1, \ldots, x_\mathsf{n}$ and sends the signed data items to the server. The server can use a special procedure Eval that, on input a program $\mathcal{P}$ and a collection of signatures $\sigma_1, \ldots, \sigma_\mathsf{n}$, outputs a signature $\sigma_{\mathcal{P},\mathsf{y}}$. Given Alice's public key and a triple $(\mathcal{P}, \mathsf{y}, \sigma_{\mathcal{P},\mathsf{y}})$, Bob (or anyone else) can get convinced that y is the correct output of $\mathcal{P}$ on inputs $(x_1, \ldots, x_\mathsf{n})$ signed by Alice. Very informally, homomorphic signatures are secure in the sense that an untrusted server (without knowing Alice's secret key) must not be able to convince the verifier of a false result. An additional property that makes this cryptographic primitive interesting and non-trivial is that signatures must be

succinct. This means that the size of $\sigma_{\mathscr{P},\mathsf{y}}$ must be significantly smaller than $\mathscr{P}$'s input size, *e.g.*, $size(\sigma_{\mathscr{P},\mathsf{y}}) = O(\log \mathsf{n})$.

The notion of homomorphic signatures was proposed by Desdmedt [16] and first formalized by Johnson *et al.* [24]. Boneh *et al.* [4] proposed the first scheme for computing linear functions over signed vectors and showed an application to preventing pollution attacks in linear network coding. Following [4], a long series of works (e.g., [20,6,1,12,13,19,2,26,8,9,15,11]) addressed the problem of constructing linearly-homomorphic signatures obtaining new schemes that improved on multiple fronts, such as efficiency, security, and privacy. A few more works addressed the problem of constructing schemes for more expressive functionalities [5,14,7,23]. Boneh and Freeman [5] proposed the first scheme for polynomial functions based on lattices, which was later improved by Catalano, Fiore and Warinschi [14] based on multilinear maps. In 2015, Gorbunov, Vaikuntanathan and Wichs [23] constructed the first HS scheme for arbitrary circuits of bounded depth from standard lattices.

Multi-Key Homomorphic Signatures. In a recent work, Fiore *et al.* [17] initiated the study of *multi-key homomorphic signatures* (MK-HS). In a nutshell, MK-HS are homomorphic signatures that allow for computing on data signed using different secret keys. This capability extends that one of previously known homomorphic signatures, and is useful in all those applications where one wants to compute on data provided (and signed) by multiple users. In addition to formally defining the notion of multi-key homomorphic signatures, Fiore *et al.* proposed a construction of MK-HS based on lattices that supports bounded depth circuits. Their scheme is obtained by extending the techniques of the single-key scheme of Gorbunov *et al.* [23]. Another recent work by Lai *et al.* [25] shows how to build an MK-HS using SNARKs and digital signatures. However, since SNARKs are likely to be based on non-falsifiable assumptions [22], the resulting MK-HS also relies on non standard assumptions.

1.1 Our Contribution

In this work, we continue the study of multi-key homomorphic signatures. Our main interest is to identify connections between multi-key homomorphic signatures and their single-key counterpart. In particular, we provide the first generic method to construct multi-key homomorphic signatures from (sufficiently expressive) single-key HS schemes. Our main contribution is a compiler, called Matrioska, that yields the following result:

Theorem 1 (Informal). *Let* HS *be a homomorphic signature scheme for circuits of polynomial size. Then, for a constant* t *representing the number of distinct keys involved in a computation, there exists a multi-key homomorphic signature scheme* $\mathsf{MKHS}(\mathsf{HS}, t)$ *for circuits of polynomial size. Furthermore, if* HS *has signatures bounded by a fixed polynomial* $p(\lambda)$*,* $\mathsf{MKHS}(\mathsf{HS}, t)$ *has signatures bounded by* $t \cdot p(\lambda)$*.*

Our result essentially shows that for a sufficiently expressive class of functions multi-key and single-key homomorphic signatures are equivalent. Our construction is the first to establish a formal connection between these two primitives without resorting to powerful primitives such as SNARKs which only yield constructions from non-falsifiable assumptions. Also, we propose a new methodology to construct MK-HS, which is the first alternative to the only known construction from standard assumptions [17]. In particular, while the techniques in [17] are specific to an algebraic lattice setting, our construction works in a generic fashion and as such it will allow to immediately obtain new MK-HS schemes from any future proposal of single-key HS.

Our MK-HS construction is quite involved and its efficiency is, admittedly, theoretical. In particular, in order to support circuits of (polynomial) size s, we need to start from a single-key HS scheme that supports circuits of size $s^{{c_s}^{t-1}}$, where t is the number of distinct keys involved in the computation and c_s is some constant that depends on the single-key HS scheme. Therefore our generic construction generates multi-key homomorphic signature schemes that can support computations among a constant number of keys (*i.e.,* users) only.

Nevertheless, our MK-HS scheme has succinct signatures that have size $t \cdot p(\lambda)$, which is non-trivial as it is independent of the total number of inputs involved in the computation. Indeed, even in the multi-key setting a trivial solution to build MK-HS from digital signatures (and even from HS) would require communication linear in the total number of inputs of a computation, i.e., $O(\mathsf{n} \cdot t)$, assuming each user provides n inputs.

An overview of our techniques. The main challenge in constructing an MK-HS scheme generically from a single-key one is to obtain a construction with succinct signatures. In particular, obtaining succinctness requires some mechanism to "compress" $\mathsf{n} \cdot t$ signatures into some information that can at most depend linearly on $\log \mathsf{n}$ and t. While single-key HS allow for compressing signatures pertaining to the same key, this property seems of no utility when one needs to compute on signatures pertaining to different keys, if nothing about their structure can be assumed.[3] To overcome this challenge, we devise a novel technique that allows us to compress $\mathsf{n} \cdot t$ signatures from t different users into t signatures; for this we show how to use the homomorphic property of the single-key HS scheme in order to inductively "prove" that the signatures of the first i users verify correctly on the corresponding inputs.

In what follows we illustrate the core idea of our technique considering, for simplicity, the two-client case $t = 2$, and assuming each users contributes to the computation with n inputs.

Let $C : \{0,1\}^{2\cdot\mathsf{n}} \to \{0,1\}$ be the circuit we wish to evaluate. Given the messages $\mathsf{m}_1, \ldots \mathsf{m}_\mathsf{n}$ by user id_1 and $\mathsf{m}_{\mathsf{n}+1}, \ldots \mathsf{m}_{2\cdot\mathsf{n}}$ by user id_2, we wish to authenticate the output of $\mathsf{y} = C(\mathsf{m}_1, \ldots, \mathsf{m}_{2\cdot\mathsf{n}})$. Let σ_i be the signature for the message m_i; in particular the first n signatures and the last n signatures are associated to different secret keys.

The initial step is to construct a $(2 \cdot \mathsf{n})$-input circuit E_0 such that $E_0(x_1, \ldots, x_{2\mathsf{n}}) = 1$ iff $C(x_1, \ldots, x_{2\mathsf{n}}) = \mathsf{y}$. Second, define a new circuit $E_1 : \{0,1\}^\mathsf{n} \to \{0,1\}$ that is E_0 with the last n inputs hardwired: $E_1(x_1, \ldots, x_\mathsf{n}) = E_0(x_1, \ldots, x_\mathsf{n}, \mathsf{m}_{\mathsf{n}+1}, \ldots, \mathsf{m}_{2\mathsf{n}})$. Now E_1 is a circuit that has inputs by a single client only, thus we can run $\hat{\sigma}_1 \leftarrow \mathsf{HS.Eval}(E_1, \mathsf{pk}_1, \sigma_1, \ldots, \sigma_\mathsf{n})$. By the correctness of the single-key homomorphic signature scheme it must hold $\mathsf{HS.Verify}(E_1, \mathsf{pk}_1, \hat{\sigma}_1, 1) = 1$. At this point, we already compressed the signatures $\sigma_1, \ldots, \sigma_\mathsf{n}$ into a single signature $\hat{\sigma}_1$. This is however not yet sufficient for succinctness because verifying $\hat{\sigma}_1$ requires the circuit E_1, which in turn requires to transmit to the verifier n messages $(\mathsf{m}_{\mathsf{n}+1}, \ldots, \mathsf{m}_{2\mathsf{n}})$ to let him reconstruct E_1.

This is where the inductive reasoning, and our new technique, begins. Very intuitively, we use the signatures of the second user to "prove" that $\mathsf{HS.Verify}(E_1, \mathsf{pk}_1, \hat{\sigma}_1, 1) = 1$, without letting the verifier run this verification explicitly. Let us see $\mathsf{H} = \mathsf{HS.Verify}((E_1, (\tau_1, \ldots, \tau_\mathsf{n})), \mathsf{pk}_1, \hat{\sigma}_1, 1)$ as a binary string with the description of a (no input) circuit. Look for the bits of H where the values $\mathsf{m}_{\mathsf{n}+1}, \ldots, \mathsf{m}_{2\mathsf{n}}$ are embedded. We can define a new circuit description E_2 that is the same as H except that the hardwired values $\mathsf{m}_{\mathsf{n}+1}, \ldots, \mathsf{m}_{2\mathsf{n}}$ are replaced with input gates. Thus E_2 is an n-input circuit satisfying $E_2(\mathsf{m}_{\mathsf{n}+1}, \ldots, \mathsf{m}_{2\mathsf{n}}) = \mathsf{HS.Verify}(E_1, \mathsf{pk}_1, \hat{\sigma}_1, 1)$, which returns 1 by correctness of HS.

[3] This is the case if one aims for a generic single-key to multi-key construction. In contrast, knowing for example the algebraic structure of signatures can be of help, as exploited in [17].

Now, the crucial observation is that E_2 is a circuit on inputs by the second client only. Thus, we can run $\hat{\sigma}_2 \leftarrow \mathsf{HS.Eval}(E_2, \mathsf{pk}_2, \sigma_{\mathsf{n}+1}, \ldots, \sigma_{2\mathsf{n}})$. By the correctness of the HS scheme, $\mathsf{HS.Verify}(E_2, \mathsf{pk}_2, \hat{\sigma}_2, 1) = 1$. Note that E_2 does not contain any of the messages $\mathsf{m}_1, \ldots, \mathsf{m}_{2\cdot\mathsf{n}}$ hardwired; in particular E_2 is completely determined by C, y, pk_1, $\hat{\sigma}_1$ and a description of $\mathsf{HS.Verify}$. Hence, given $(\hat{\sigma}_1, \hat{\sigma}_2)$ the verifier can reconstruct E_2 and check if $\mathsf{HS.Verify}(E_2, \mathsf{pk}_2, \hat{\sigma}_2, 1) = 1$. Intuitively, this proves that for some messages signed by the second user $E_2(\mathsf{m}_{\mathsf{n}+1}, \ldots, \mathsf{m}_{2\mathsf{n}}) = 1$. By the correctness of HS, this in turn implies $E_1(\mathsf{m}_1, \ldots, \mathsf{m}_{\mathsf{n}}) = 1$ for some messages signed by the first user; and by construction of E_1 the latter implies $C(\mathsf{m}_1, \ldots, \mathsf{m}_{2\mathsf{n}}) = \mathsf{y}$.

Our compiler, extends the above idea to multiple users, showing that at each step i the problem consists in proving correctness of a computation E_{i-1} that depends only on the inputs of user i, while inputs of users $> i$ are hardwired into it. This means that a progressive application of this idea lets the hardwired inputs progressively disappear up to the point of obtaining a circuit E_t which has no input hardwired and thus can be reconstructed by the verifier. This is the only computation explicitly checked by the verifier. By construction, E_t encodes the nested execution of several single-key HS verifications (from which our compiler's name "Matrioska"), and validity of E_t implicitly implies that each E_i returns 1 (even if the verifier does not know E_i itself). In this description we favor intuition to precision. In fact, the full development of this technique is given in Section 3 and requires to take care of several details to ensure that the verifier can reconstruct the last circuit without any knowledge of the input messages.

2 Preliminaries

Notation. The security parameter of our schemes is denoted by λ. For any $n \in \mathbb{N}$, we use $[n]$ to denote the set $[n] := \{1, \ldots, n\}$. The symbol lg denotes the logarithm in base 2; $||$ denotes the string concatenation, *e.g.,* $(00)||(10) = (0010)$; bold font letters, *e.g.,* $\boldsymbol{\sigma} = (\sigma_1, \ldots, \sigma_{\mathsf{n}})$, denote vectors. A function $\epsilon(\lambda)$ is said negligible in λ (denoted as $\epsilon(\lambda) = \mathsf{negl}(\lambda)$) if $\epsilon(\lambda) = O(\lambda^{-c})$ for every constant $c > 0$. Also, we often write $\mathsf{poly}(\cdot)$ to denote a function that can be expressed as a polynomial.

2.1 Circuits

We use a modeling of circuits similar to the one in [3]. We define circuits as 6-tuples $C = (\mathsf{n}, \mathsf{u}, \mathsf{q}, \mathsf{L}, \mathsf{R}, \mathsf{G})$. The value $\mathsf{n} \geq 1$ denotes the number of inputs to the circuit, $\mathsf{u} \geq 1$ is the number of outputs and $\mathsf{q} \geq 1$ is the number of gates. Let w denote the total number of wires in the circuit. For the circuits considered in this work $\mathsf{w} = \mathsf{n}+\mathsf{q}$. The functions L and R define respectively the left and right input wire to any given gate $g \in [\mathsf{q}]$, formally, $\mathsf{L}, \mathsf{R} : [\mathsf{q}] \to [\mathsf{w}] \cup \{0\}$. Finally, $\mathsf{G} : [\mathsf{q}] \to \{0, 1\}$ encodes the gates by mapping each gate $g \in [\mathsf{q}]$ into a single bit G_g. In our construction we treat circuit descriptions C as binary strings. Similarly to [3], the size of our circuit description is quasi-linear in the number of wires: $|C| \in O(\mathsf{w}\lg(\mathsf{w}))$. Differently from [3], we number gates from 1 to q (instead of from $\mathsf{n}+1$ to $\mathsf{n}+\mathsf{q}$) and label the outgoing wire of a gate g as $g + \mathsf{n}$. Moreover, we introduce the 0 wire to denote *constant output* gates, *e.g.,* no-input gates or gates that have the same output independently of the input values, and allow for a gate to have the same left and right input, *i.e.,* $\mathsf{L}(g) \leq \mathsf{R}(g) < g + \mathsf{n}$. The largest component in the string C is the descriptions of the function L (and R), that is a sequence of q values in $[\mathsf{w}] \cup \{0\}$, therefore $|\mathsf{L}| = |\mathsf{R}| = \mathsf{q}\lg(\mathsf{w}+1)$. Hence, for a fixed and reasonable encoding it holds $|C| \in O(\mathsf{w}\lg(\mathsf{w}))$.

As an example of a circuit consider the following EQ^{y} circuit (that will be used in our generic compiler)

Definition 1 (The equal-to circuit EQ^{y}). *We define EQ^{y} to be the circuit that, for a given value $\mathsf{y} \in \{0,1\}$, returns 1 if the input bit equals y, 0 otherwise. Formally,*

$$EQ^{\mathsf{y}} = \big(1, 1, 5, (01134), (02325), (\mathsf{y}, 1, 1, 1, 1)\big) .$$

A representation of EQ^{y} is given in Figure 1. $EQ^{\mathsf{y}} = \big(1, 1, 5, (01134), (02325), (\mathsf{y}, 1, 1, 1, 1)\big)$.

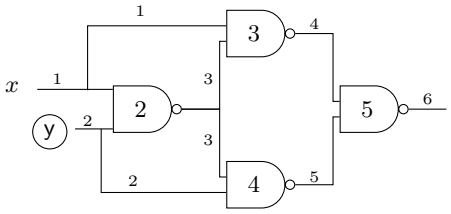

Fig. 1. The EQ^{y} circuit for the value $\mathsf{y} = \mathsf{m}$.

Evaluating a Circuit. In order to evaluate a circuit on a given input, we need a gate-functionality function that translates each bit in (the description of) G into the output of a gate. Let $\mathsf{G}(g)$ denote the g-th bit in the description of G, we define the gate-functionality function $\gamma : [\mathsf{q}] \times \{0,1\}^2 \to \{0,1\}$ as:

$$\gamma(g, x_l, x_r) = \begin{cases} \mathsf{G}(g) & \text{if } \mathsf{L}(g) = 0, \\ x_l & \text{if } (\mathsf{L}(g) \neq 0 \text{ and } \mathsf{G}(g) = 0), \\ \mathsf{NAND}(x_l, x_r) & \text{if } (\mathsf{L}(g) \neq 0 \text{ and } \mathsf{G}(g) = 1) \end{cases}$$

Note that when $\mathsf{L}(g) = 0$ the gate-functionality of g is the constant-gate that always returns the value $\mathsf{G}(g) \in \{0,1\}$. Otherwise, g is a *proper* gate: if $\mathsf{G}(g) = 0$ it returns the left input to g, while if $\mathsf{G}(g) = 1$ it returns the NAND between the two input values.

We define the evaluation function $\mathsf{ev}_{\mathsf{circ}}$ on a circuit $C = (\mathsf{n}, \mathsf{u}, \mathsf{q}, \mathsf{L}, \mathsf{R}, \mathsf{G})$ and an n-bit string $(x_1, x_2, \ldots, x_{\mathsf{n}})$ as:

process $\mathsf{ev}_{\mathsf{circ}}(C, (x_1, \ldots, x_{\mathsf{n}}))$
  for g from 1 to q **do**:
    $l \leftarrow \mathsf{L}(g); r \leftarrow \mathsf{R}(g); x_g \leftarrow \gamma(g, x_l, x_r);$
  return $(x_{\mathsf{n}+\mathsf{q}-\mathsf{u}+1}, \ldots, x_{\mathsf{n}+\mathsf{q}})$.

We will often shorten $\mathsf{ev}_{\mathsf{circ}}(C, (x_1, \ldots, x_{\mathsf{n}}))$ into $C(x_1, \ldots, x_{\mathsf{n}})$.

Sequential composition of circuits. Given two circuits, C_1 and C_2, we say that C_1 is *composable* with C_2 if $\mathsf{u}_1 = \mathsf{n}_2$. Intuitively, composition connects each output wire of C_1 with one input wire of C_2. We denote the circuit composition as $C_3 = C_1 \triangleright C_2$. The resulting circuit $C_3 = (\mathsf{n}_3, \mathsf{u}_3, \mathsf{q}_3, \mathsf{L}_3, \mathsf{R}_3, \mathsf{G}_3)$ is defined as: $\mathsf{n}_3 = \mathsf{n}_1$, $\mathsf{u}_3 = \mathsf{u}_2$, $\mathsf{q}_3 = \mathsf{q}_1 + \mathsf{q}_2$. Let w_i be the number of wires in C_i, then

$$\mathsf{L}_3 = \begin{cases} \mathsf{L}_1(i) & \text{for } i \in [\mathsf{w}_1] \\ 0 & \text{for } i \in [\mathsf{w}_1 + \mathsf{w}_2] \setminus [\mathsf{w}_1] \text{ and } \mathsf{L}_2(i - \mathsf{w}_1) = 0 \\ \mathsf{L}_2(i - \mathsf{w}_1) + \mathsf{w}_1 - \mathsf{u}_1 & \text{for } i \in [\mathsf{w}_1 + \mathsf{w}_2] \setminus [\mathsf{w}_1] \text{ and } \mathsf{L}_2(i - \mathsf{w}_1) \neq 0 \end{cases}$$

Note that the entries of L_3 that are set to 0 preserve constant output gates. The right-input function R_3 is defined analogously. The right-input function R_3 is defined analogously. Finally, $\mathsf{G}_3 = \mathsf{G}_1 || \mathsf{G}_2$.

Note that any circuit of the form $C = (\mathsf{n}, 1, \mathsf{q}, \mathsf{L}, \mathsf{R}, \mathsf{G})$ is composable with EQ^{y} and the composed circuit

$$\begin{aligned} D &= C \triangleright EQ^{\mathsf{y}} \\ &= (\mathsf{n}, 1, \mathsf{q}+5, \mathsf{L}||(0, \mathsf{w}, \mathsf{w}, \mathsf{w}+2, \mathsf{w}+3), \mathsf{R}||(0, \mathsf{w}+1, \mathsf{w}+2, \mathsf{w}+1, \mathsf{w}+4), \mathsf{G}||(\mathsf{y}, 1, 1, 1, 1)) \end{aligned}$$

2.2 Multi-Key Homomorphic Signatures

We start by recalling the notion of labeled programs of Gennaro and Wichs [21].

Labeled Programs [21]. A labeled program $\mathscr{P}$ is a tuple $(C, \ell_1, \ldots, \ell_t)$, such that $C : \mathcal{M}^t \to \mathcal{M}$ is a function of t variables (*e.g.,* a circuit) and $\ell_i \in \{0,1\}^*$ is a label for the i-th input of C. Labeled programs can be composed as follows: given $\mathscr{P}_1, \ldots, \mathscr{P}_n$ and a function $G : \mathcal{M}^n \to \mathcal{M}$, the composed program $\mathscr{P}^*$ is the one obtained by evaluating G on the outputs of $\mathscr{P}_1, \ldots, \mathscr{P}_n$, and it is denoted as $\mathscr{P}^* = G(\mathscr{P}_1, \ldots, \mathscr{P}_n)$. The labeled inputs of $\mathscr{P}^*$ are all the distinct labeled inputs of $\mathscr{P}_1, \ldots, \mathscr{P}_n$ (all the inputs with the same label are grouped together and considered as a unique input of $\mathscr{P}^*$).

We recall the definitions of Fiore *et al.* [17] for multi-key homomorphic authenticators, adapted to the case of signature schemes only. Intuitively, MKHS extend the existing notion of homomorphic signatures in such a way that the evaluation procedure is correct on computations over data signed by different secret keys. Following [17], we consider labels where $\ell = (\mathsf{id}, \tau)$, such that id is a given client identity and τ is a tag which refers to the client's input data. To ease the reading, we use the compact and improper notation $\mathsf{id} \in \mathscr{P}$ meaning that there exists at least one index label ℓ in the description of $\mathscr{P} = (C, (\ell_1, \ldots, \ell_{\mathsf{n}}))$ such that $\ell = (\mathsf{id}, \tau)$ for some string τ.

Definition 2 (Multi-Key Homomorphic Signature [17]). *A multi-key homomorphic signature scheme* MKHS *is a tuple of five PPT algorithms* MKHS = (MKHS.Setup, MKHS.KeyGen, MKHS.Sign, MKHS.Eval, MKHS.Verify) *that satisfies the properties of* authentication correctness, evaluation correctness, succinctness *and* security. *The algorithms are defined as follows:*

MKHS.Setup(1^λ). *The setup algorithm takes as input the security parameter λ and outputs some public parameters* pp *including a description of an identity space* ID, *a tag space $\mathcal{T}$ (these implicitly define the label space $\mathcal{L} = \mathsf{ID} \times \mathcal{T}$), a message space $\mathcal{M}$ and a set of admissible functions $\mathcal{F}$. The* pp *are input to all the following algorithms, even when not specified.*

MKHS.KeyGen(pp). *The key generation algorithm takes as input the public parameters and outputs a pair of keys* (sk, pk), *where* sk *is a secret signing key, while* pk *is the public evaluation and verification key.*

MKHS.Sign$(\mathsf{sk}, \Delta, \ell, \mathsf{m})$. *The sign algorithm takes as input a secret key* sk, *a dataset identifier Δ, a label $\ell = (\mathsf{id}, \tau)$ for the message* m, *and it outputs a signature σ.*

MKHS.Eval$(\mathscr{P}, \Delta, \{(\sigma_i, \mathsf{pk}_{\mathsf{id}_i})\}_{i\in[\mathsf{n}]})$. *The evaluation algorithm takes as input a labeled program $\mathscr{P} = (C, (\ell_1, \ldots, \ell_{\mathsf{n}}))$, where C is an* n*-input circuit $C : \mathcal{M}^{\mathsf{n}} \longrightarrow \mathcal{M}$, a dataset identifier Δ and a set of signature and public-key pairs $\{(\sigma_i, \mathsf{pk}_{\mathsf{id}_i})\}_{i\in[\mathsf{n}]}$. The output is an homomorphic signature σ.*

MKHS.Verify$(\mathscr{P}, \Delta, \{\mathsf{pk}_{\mathsf{id}}\}_{\mathsf{id}\in\mathscr{P}}, \mathsf{m}, \sigma)$. *The verification algorithm takes as input a labeled program $\mathscr{P} = (C, (\ell_1, \ldots, \ell_{\mathsf{n}}))$, a dataset identifier Δ, the set of public keys $\{\mathsf{pk}_{\mathsf{id}}\}_{\mathsf{id}\in\mathscr{P}}$ corresponding to those identities* id *involved in the program $\mathscr{P}$, a message* m *and an homomorphic signature σ. It outputs* 0 *(reject) or* 1 *(accept).*

Remark 1 (Single/Multi-Hop Evaluation). Similarly to fully homomorphic encryption, we call a (multi-key) homomorphic signature i-Hop if the Eval algorithm can be executed on its own outputs up to i times. We call *single-hop* a scheme where Eval can be executed only on fresh signatures, i.e., generated by Sign, whereas a multi-hop scheme is a scheme that is i-Hop for all i.

Authentication Correctness. A multi-key homomorphic signature satisfies authentication correctness if for all public parameters $\mathsf{pp} \leftarrow \mathsf{MKHS.Setup}(1^\lambda)$, any key pair $(\mathsf{sk_{id}}, \mathsf{pk_{id}}) \leftarrow \mathsf{MKHS.KeyGen}(\mathsf{pp})$, any dataset identifier Δ, any label $\ell = (\mathsf{id}, \tau) \in \mathscr{L}$, any message $\mathsf{m} \in \mathcal{M}$ and any signature $\sigma \leftarrow \mathsf{MKHS.Sign}(\mathsf{sk}, \Delta, \ell, \mathsf{m})$, it holds that

$$\Pr\left[\mathsf{MKHS.Verify}(\mathcal{I}_\ell, \Delta, \mathsf{pk}, \mathsf{m}, \sigma) = 1\right] \geq 1 - \mathsf{negl}\ .$$

Evaluation Correctness. A multi-key homomorphic signature satisfies evaluation correctness if

$$\Pr\left[\mathsf{MKHS.Verify}(\mathscr{P}', \Delta, \{\mathsf{pk_{id}}\}_{\mathsf{id}\in\mathscr{P}'}, \mathsf{m}', \sigma') = 1\right] \geq 1 - \mathsf{negl}$$

where the equality holds for a fixed description of the public parameters $\mathsf{pp} \leftarrow \mathsf{MKHS.Setup}(1^\lambda)$, an arbitrary set of honestly generated keys $\{(\mathsf{sk_{id}}, \mathsf{pk_{id}})\}_{\mathsf{id}\in\tilde{\mathsf{ID}}}$ for some $\tilde{\mathsf{ID}} \subseteq \mathsf{ID}$, with $|\tilde{ID}| = t$, a dataset identifier Δ, a function $C : \mathcal{M}^{\mathsf{n}} \to \mathcal{M}$, and any set of program/message/signature triples $\{(\mathscr{P}_i, \mathsf{m}_i, \sigma_i)\}_{i\in[\mathsf{n}]}$ such that $\mathsf{MKHS.Verify}(\mathscr{P}_i, \Delta, \{\mathsf{pk_{id}}\}_{\mathsf{id}\in\mathscr{P}_i}, \mathsf{m}_i, \sigma_i) = 1$ for all $i \in [\mathsf{n}]$, and $\mathsf{m}' = g(\mathsf{m}_1, \ldots, \mathsf{m}_\mathsf{n})$, $\mathscr{P}' = g(\mathscr{P}_1, \ldots, \mathscr{P}_\mathsf{n})$, and $\sigma' = \mathsf{Eval}(C, \{(\sigma_i, PK_i)\}_{i\in[n]})$ where $PK_i = \{\mathsf{pk_{id}}\}_{\mathsf{id}\in\mathscr{P}_i}$.

Succinctness. Succinctness is one of the crucial properties that make multi-key homomorphic signatures an interesting primitive. Intuitively, a MKHS scheme is succinct if the size of every signature depends only logarithmically on the size of a dataset. More formally, let $\mathsf{pp} \leftarrow \mathsf{MKHS.Setup}(1^\lambda)$, $\mathscr{P} = (C, (\ell_1, \ldots, \ell_\mathsf{n}))$ with $\ell_i = (\mathsf{id}_i, \tau_i)$, $(\mathsf{sk_{id}}, \mathsf{pk_{id}}) \leftarrow \mathsf{MKHS.KeyGen}(\mathsf{pp})$ for all $\mathsf{id} \in [n]$. and $\sigma_i \leftarrow \mathsf{MKHS.Sign}(\mathsf{sk}_{\mathsf{id}_i}, \Delta, \ell_i, \mathsf{m}_i)$, for all $i \in [\mathsf{n}]$, then MKHS has succinct signatures if there exists a fixed polynomial $\mathsf{poly}(\cdot)$ such that $size(\sigma) = \mathsf{poly}(\lambda, t, \log \mathsf{n})$ where $\sigma = \mathsf{MKHS.Eval}(\mathscr{P}, \{(\sigma_i, \mathsf{pk}_{\mathsf{id}_i})\}_{i\in[n]})$.

Security. We adopt Fiore *et al.*'s security model [17]. Very intuitively, a multi-key homomorphic signature scheme is secure if the adversary, who can request to multiple users signatures on messages of its choice, can produce only signatures that are either the ones it received, or ones that are obtained by correctly executing the Eval algorithm. In addition, in the multi-key setting the adversary is also allowed to corrupt users but this shall not affect the integrity of computations performed on data signed by other (un-corrupted) users of the system.

Formally, the security is defined via the $\mathsf{MK\text{-}HomUF\text{-}CMA}$ security experiment below.

Setup. The challenger $\mathcal{C}$ runs $\mathsf{MKHS.Setup}(1^\lambda)$ and sends the output public parameters pp to the adversary $\mathcal{A}$.

Sign Queries. The adversary can adaptively submit queries of the form $(\Delta, \ell, \mathsf{m})$, where Δ is a dataset identifier, $\ell = (\mathsf{id}, \tau)$ is a label in $\mathsf{ID} \times \mathcal{T}$ and $\mathsf{m} \in \mathcal{M}$ is a message. The challenger answers the queriesperforming all the 1-4 checks below:

1. If (ℓ, m) is the first query for the dataset Δ, the challenger initializes an empty list $L_\Delta = \emptyset$.
2. If $(\Delta, \ell, \mathsf{m})$ is the first query with identity id, the challenger generates the keys for that identity: $(\mathsf{sk_{id}}, \mathsf{pk_{id}}) \leftarrow \mathsf{KeyGen}(\mathsf{pp})$. and proceeds to step 3.
3. If $(\Delta, \ell, \mathsf{m})$ is such that $(\ell, \mathsf{m}) \notin L_\Delta$, the challenger computes $\sigma \leftarrow \mathsf{MKHS.Sign}(\mathsf{sk_{id}}, \Delta, \ell, \mathsf{m})$ (this is possible since $\mathcal{C}$ has already generated the keys for the identity id). Then the challenger updates the list $L_\Delta \leftarrow L_\Delta \cup (\ell, \mathsf{m})$ and returns $(\sigma, \mathsf{pk_{id}})$ to $\mathcal{A}$.

4. If $(\Delta, \ell, \mathsf{m})$ is such that $(\ell, \cdot) \notin L_\Delta$, that is, the adversary had already made a query $(\Delta, \ell, \mathsf{m}')$ for some message m', the challenger ignores the query. Note that this means thatfor a given (Δ, ℓ) pair only one message can be obtained.

Corruption Queries. At the beginning of the game, the challenger initialises an empty list $\mathsf{L_{corr}} = \emptyset$ of corrupted identities. During the game, the adversary can adaptively perform corruption queries by sending $\mathsf{id} \in \mathsf{ID}$ to the challenger. If $\mathsf{id} \notin \mathsf{L_{corr}}$ the challenger updates the list $\mathsf{L_{corr}} \leftarrow \mathsf{L_{corr}} \cup \mathsf{id}$ and answers the query with the pair $(\mathsf{sk_{id}}, \mathsf{pk_{id}})$ generated using KeyGen (if not done before). If $\mathsf{id} \in \mathsf{L_{corr}}$ the challenger replies with keys $(\mathsf{sk_{id}}, \mathsf{pk_{id}})$ assigned to id before.

Forgery. At the end of the game, the adversaryoutputs a tuple $(\mathcal{P}^*, \Delta^*, \{\mathsf{pk}^*_{\mathsf{id}}\}_{\mathsf{id} \in \mathcal{P}^*}, \mathsf{y}^*, \sigma^*)$. The experiment outputs 1 if the tuple returned by $\mathcal{A}$ is a forgery (defined below), and 0 otherwise.

A MK-HS scheme MKHS is *unforgeable* if for every PPT adversary $\mathcal{A}$, its advantage $\mathrm{Adv}^{\mathsf{MKHS}}_{\mathcal{A}}(\lambda) = \Pr[\mathsf{MK\text{-}HomUF\text{-}CMA}_{\mathcal{A},\mathsf{MKHS}}(\lambda) = 1]$ is $\mathsf{negl}(\lambda)$.

Definition 3 (Forgery). *We consider an execution of* $\mathsf{MK\text{-}HomUF\text{-}CMA}$ *where* $(\mathcal{P}^*, \Delta^*, \{\mathsf{pk}^*_{\mathsf{id}}\}_{\mathsf{id} \in \mathcal{P}^*}, \mathsf{y}^*, \sigma^*)$ *is the tuple returned by* $\mathcal{A}$ *at the end of the experiment. Let* $\mathcal{P}^* = (C^*, \ell^*_1, \ldots, \ell^*_{\mathsf{n}})$*. The adversary's output is said to be a successful forgery against the multi-key homomorphic signature scheme if:* $\mathsf{MKHS.Verify}(\mathcal{P}^*, \Delta^*, \{\mathsf{pk}^*_{\mathsf{id}}\}_{\mathsf{id} \in \mathcal{P}^*}, \mathsf{y}^*, \sigma^*) = 1$ *and at least one of the following conditions hold:*

Type-1 forgery*: the dataset* Δ^* *was never initialised.*

Type-2 forgery*: for all* $\mathsf{id} \in \mathcal{P}^*$, $\mathsf{id} \notin \mathsf{L_{corr}}$ *and* $(\ell^*_i, \mathsf{m}_i) \in L_{\Delta^*}$ *for all* $i \in [\mathsf{n}]$*, but* $\mathsf{y}^* \neq C^*(\mathsf{m}_1, \ldots, \mathsf{m}_{\mathsf{n}})$.

Type-3 forgery*: there exists (at least) one index* $i \in [\mathsf{n}]$ *such that* ℓ^*_i *was never queried, i.e.,* $(\ell^*_i, \cdot) \notin L_{\Delta^*}$ *and* $\mathsf{id}_i \notin \mathsf{L_{corr}}$ *is a non-corrupted identity.*

Non-adaptive corruption queries. We also recall a proposition given in [17], which shows that it is sufficient to prove security for *non-adaptive* corruption queries. This is a setting where the adversary $\mathcal{A}$ can perform corruption queries only on identities for which no signature query had already been performed. This proposition can be used to simplify security proofs.

Proposition 1 ([17]). MKHS *is secure against adversaries that do not make corruption queries if and only if* MKHS *is secure against adversaries that make non-adaptive corruption queries.*

2.3 Homomorphic Signatures

Despite some minor syntactic modifications, homomorphic signatures can be seen as a special case of multi-key homomorphic signatures for algorithms that run on inputs by a single user only. For the purpose of this work, single-key homomorphic signature schemes are defined by five PPT algorithms $\mathsf{HS} = (\mathsf{HS.Setup}, \mathsf{HS.KeyGen}, \mathsf{HS.Sign}, \mathsf{HS.Eval}, \mathsf{HS.Verify})$ that have the same input-output behavior as the corresponding algorithms in MKHS except:

- There is no identity space ID and the labels are simply $\ell = \tau$.

- The evaluation algorithm $\mathsf{HS.Eval}$ takes as input a circuit C, a single public key pk and a set of signatures $\sigma_1, \ldots, \sigma_{\mathsf{n}}$. In particular $\mathsf{HS.Eval}$ runs without labels or dataset identifier.

- The verification algorithm $\mathsf{HS.Verify}$ accepts inputs from a single user only, *i.e.,* the labeled program $\mathcal{P}$ is of the form $\mathcal{P} = (C, (\tau_1, \ldots, \tau_{\mathsf{n}}))$ and only one public key pk is provided.

The properties of authentication and evaluation correctness are analogous to the ones for MKHS in the case of computations on inputs by a single client. Regarding succinctness, a homomorphic signature scheme HS has *succinct signatures* if the size of any signature σ output by $\mathsf{HS.Eval}$ depends only logarithmic in the number n inputs to the labelled program, *i.e.,* $size(\sigma) = \mathsf{poly}(\lambda, \log(\mathsf{n}))$.

Finally, we observe that the specialization to the single-key setting of the above security definition corresponds to the strong-adaptive security definition of HS that is formalized in [10]. In particular, the definitions in [10] allow for a simple treatment of Type-3 forgeries. In [10] it is also shown that HS constructions for circuits that are secure in this stronger model can be generically built, e.g., from [23].

3 The Matrioska compiler

In this section, we present Matrioska: a generic compiler from a single-key homomorphic signature scheme $\mathsf{HS} = (\mathsf{HS.KeyGen}, \mathsf{HS.Sign}, \mathsf{HS.Eval}, \mathsf{HS.Verify})$ to a (single-hop) multi-key scheme $\mathsf{MKHS} = (\mathsf{MKHS.KeyGen}, \mathsf{MKHS.Sign}, \mathsf{MKHS.Eval}, \mathsf{MKHS.Verify})$.

Theorem 2. *Let* HS *be a homomorphic signature scheme that is correct and unforgeable. Then, for any given integer number* $\mathsf{T} \geq 1$ *there exists a multi-key homomorphic signature scheme* $\mathsf{MKHS}(\mathsf{HS}, \mathsf{T})$ *that supports computations on signatures generated using at most* T *distinct keys, it is correct and unforgeable. Furthermore, if* HS *supports circuits of maximum size* s *and maximum depth* d *and it has succinctness* l*, then* $\mathsf{MKHS}(\mathsf{HS}, \mathsf{T})$ *on* T *distinct users has succinctness* $\mathsf{T} \cdot l$*, and can support circuits of size* s' *and depth* d' *provided that* $s > (s')^{{c_s}^{\mathsf{T}-1}}$ *and* $d > \max\{d', d_{\mathsf{HSV}}((s')^{{c_s}^{\mathsf{T}-1}}, \lambda)\}$*, where* d_{HSV} *and* c_s *are a function and a non-negative constant that depend from the single-key scheme* HS*.*

More precisely, d_{HSV} expresses the depth of the circuit for the verification algorithm $\mathsf{HS.Verify}$ as a function of its input length (which includes the description of the labeled program $\mathscr{P}$); c_s is a constant such that the size of $\mathsf{HS.Verify}$ on input a circuit C is $size(C)^{c_s}$. Notice that by efficiency of HS (*i.e.,* its algorithms are polynomial time), such c_s exists, and d_{HSV} can, in the worst case, alsobe written as $size(C)^{c_d}$ for some other constant c_d.

Theorem 2 can be instantiated in two ways. If HS is a fully-homomorphic signature (whose existence is not yet known), then for any $s' = \mathsf{poly}(\lambda)$ and for any constant number T, we are guaranteed that HS is executed on poly-sized circuits. Otherwise, if HS is an HS for circuits of bounded polynomial depth (and of any, or bounded, polynomial size), as *e.g.,* [23], then for any $s' = \mathsf{poly}(\lambda)$ and for any fixed number of keys T, we can derive a polynomial bound d on the depth. Let us now turn to the proof of Theorem 2. This is constructive. First we show a method to define MKHS given a HS scheme and a value T. Next, in a sequence of lemmas, we prove all the properties stated in the theorem.

Our construction is rather involved. Therefore, to help the reader,in the next section we first illustrate our ideas for a simple case of a computation that takes inputs from three different users, and then, in Section 3.2, we describe the full compiler.

3.1 An intuition: the three-client case

We provide here a simplified example to explain the core idea of our Matrioska compiler. To ease the exposition we consider the case $t = 3$ (three clients with identities $\mathsf{id}_1, \mathsf{id}_2$ and id_3) and deliberately remove dataset identifiers. A detailed description for $t = \mathsf{n} = 3$ can be found in the Appendix A.

Let $\mathscr{P} = (C, (\ell_1, \ldots, \ell_{\mathsf{n}}))$ be a labelled program, where C a (n)-input circuit (with $\mathsf{n} = \mathsf{n}_1 + \mathsf{n}_2 + \mathsf{n}_3$) and the labels $\ell_i = (\mathsf{id}_i, \tau_i)$ are ordered, *i.e.,* first n_1 inputs belong to client id_1, the subsequent n_2

to id_2 and the last n_3 inputs to id_3. Let σ_i be the signature on message m_i for the label ℓ_i. For simplicity assume that $C(\mathsf{m}_1, \ldots, \mathsf{m}_\mathsf{n}) = \mathsf{y} = 1$.

Step 1. We want extract from C a circuit that contains only inputs by clients id_2 and id_3. To this end, we define E_1 as the partial evaluation of C on the messages $\mathsf{m}_{\mathsf{n}_1+1}, \ldots, \mathsf{m}_\mathsf{n}$. Thus, E_1 is an n_1-input circuit with hardwired in it the inputs by clients id_2 and id_3. In our framework E_1 is obtained with two basic operations on the bit string C: (1) setting any gate g with left or right input wire in $[\mathsf{n}] \setminus [\mathsf{n}_1]$ to be a constant gate (*i.e.,* setting the bits $\mathsf{L}(g)$ and $\mathsf{R}(g)$ to 0), and (2) initializing the now constant gate to the value m_i for $i \in [\mathsf{n}] \setminus [\mathsf{n}_1]$. At this point we obtained a circuit with inputs of a single client only, and we can run $\hat{\sigma}_1 \leftarrow \mathsf{HS.Eval}(E_1, \mathsf{pk}_{\mathsf{id}_1}, \sigma_1, \ldots, \sigma_{\mathsf{n}_1})$. By construction $E_1(\mathsf{m}_1, \ldots, \mathsf{m}_{\mathsf{n}_1}) = C(\mathsf{m}_1, \ldots, \mathsf{m}_\mathsf{n}) = 1$, therefore $\mathsf{HS.Verify}((E_1, (\tau_1, \ldots, \tau_{\mathsf{n}_1})), \mathsf{pk}_{\mathsf{id}_1}, \hat{\sigma}_1, 1) = 1$.

Step 2. The actual inductive procedure begins now. We wish to verify the correctness of $\hat{\sigma}_1$ using the messages input by client id_2 as variables. Consider the input to the (single-client) verification as the string $S_1 = ((E_1, (\tau_1, \ldots, \tau_{\mathsf{n}_1})), \mathsf{pk}_{\mathsf{id}_1}, \hat{\sigma}_1, 1)$. Recall that to construct the circuit E_1 we used the messages $\mathsf{m}_{\mathsf{n}_1+1}, \ldots \mathsf{m}_\mathsf{n}$ (hard-wired in its gate description). To free the inputs by client id_2 we modify S_1 in the following way: (1) identify the gates that contain the messages $\mathsf{m}_{\mathsf{n}_1+1}, \ldots, \mathsf{m}_{\mathsf{n}_1+\mathsf{n}_2}$, (2) turn these gates into input gates by setting the left/right wires to the opportune values w (using $\mathscr{P}$). Let us (formally)consider $\mathsf{HS.Verify}$ on the modified string S_1, this is a proper circuit E_2 such that $E_2(\mathsf{m}_{\mathsf{n}_1+1}, \ldots, \mathsf{m}_{\mathsf{n}_1+\mathsf{n}_2}) = \mathsf{HS.Verify}((E_1, (\tau_1, \ldots, \tau_{\mathsf{n}_1})), \mathsf{pk}_{\mathsf{id}_1}, \hat{\sigma}_1, 1) = 1$. Being E_2 a single-client circuit we can run $\hat{\sigma}_2 \leftarrow \mathsf{HS.Eval}(E_2, \mathsf{pk}_{\mathsf{id}_2}, \sigma_{\mathsf{n}_1+1}, \ldots, \sigma_{\mathsf{n}_1+\mathsf{n}_2})$.

Step 3. This is analogous to **Step 2**: we wish to verify the correctness of $\hat{\sigma}_2$ using the messages input by client id_3 as variables and define a circuit that is completely determined by public values, no hard-wired message value. Let $S_2 = ((E_2, (\tau_{\mathsf{n}_1+1}, \ldots, \tau_{\mathsf{n}_1+\mathsf{n}_2})), \mathsf{pk}_{\mathsf{id}_2}, \hat{\sigma}_2, 1)$, we free the inputs by client id_3 as in **Step 2**. We define E_3 as the formal evaluation of $\mathsf{HS.Verify}$ on the modified string S_2. By construction it holds that $E_3(\mathsf{m}_{\mathsf{n}_1+\mathsf{n}_2+1}, \ldots, \mathsf{m}_\mathsf{n}) = \mathsf{HS.Verify}((E_2, (\tau_{\mathsf{n}_1+1}, \ldots, \tau_{\mathsf{n}_1+\mathsf{n}_2})), \mathsf{pk}_{\mathsf{id}_2}, \hat{\sigma}_2, 1) = 1$, and we can run $\hat{\sigma}_3 \leftarrow \mathsf{HS.Eval}(E_3, \mathsf{pk}_{\mathsf{id}_3}, \sigma_{\mathsf{n}_1+\mathsf{n}_2+1}, \ldots, \sigma_\mathsf{n})$.

The multi-key homomorphic evaluation algorithm outputs $\hat{\sigma} = (\hat{\sigma}_1, \hat{\sigma}_2, \hat{\sigma}_3)$.
The $\mathsf{Matrioska}$ verification procedure needs only reconstruct the final circuit E_3, as this is fully determined by the public values $(\mathscr{P}, \mathsf{pk}_{\mathsf{id}_1}, \mathsf{pk}_{\mathsf{id}_2}, \hat{\sigma}_1, \hat{\sigma}_2, \mathsf{HS.Verify}, 1)$. Let $\mathscr{E}_3 = (E_3, (\tau_{\mathsf{n}_1+\mathsf{n}_2+1}, \ldots, \tau_\mathsf{n}))$, the verification concludes by running the single-key verification algorithm: $\mathsf{HS.Verify}(\mathscr{E}_3, \mathsf{pk}_3, \hat{\sigma}_3, 1)$.

3.2 The Matrioska Compiler

In this section we describe our compiler in the general case of computing on signatures generated by t different keys.

Definition 4 (Matrioska). *Let* $\mathsf{HS} = (\mathsf{HS.Setup}, \mathsf{HS.KeyGen}, \mathsf{HS.Sign}, \mathsf{HS.Eval}, \mathsf{HS.Verify})$ *be a single-key homomorphic signature scheme, we define a multi-key homomorphic signature scheme* MKHS *as follows:*

$\mathsf{MKHS.Setup}(1^\lambda, \mathsf{T}, s', d') \rightarrow \mathsf{pp}$. *The set-up algorithm takes as input the security parameter λ, a positive integer* T *that represents a bound for the maximal number of distinct identities involved in the same homomorphic computation, and bounds $s', d' = \mathsf{poly}(\lambda)$ on the size and depth respectively of the circuits used in the* MKHS.Eval *and* MKHS.Verify *algorithms. Setup first uses T, s', d' to derive two integers s and d such that $s > (s')^{c_s{}^{\mathsf{T}-1}}$ and $d > \max\{d', d_{\mathsf{HSV}}((s')^{c_s{}^{\mathsf{T}-1}}, \lambda)\}$. Next, it runs $\mathsf{HS.Setup}(1^\lambda, s, d)$ to obtain a tag space $\mathcal{T}$ (which corresponds to the label space of* HS*), a*

message space $\mathcal{M}$ and a set of admissible circuits $\mathcal{F}$.[4] *Labels of the multi-key scheme are defined as pairs $\ell = (\mathsf{id}, \tau) \in \mathsf{ID} \times \mathcal{T}$, where the first entry is a client-identity identifier. Labeled programs are of the form $\mathcal{P} = (C, (\ell_1, ..., \ell_t))$ with labels as above.*

$\mathsf{MKHS.KeyGen}(\mathsf{pp}) \to (\mathsf{pk}, \mathsf{sk})$. *The multi-key key-generation algorithm runs* $\mathsf{HS.KeyGen}$ *to obtain a public-secret key pair. This key-pair will be associated to an identity $\mathsf{id} \in \mathsf{ID}$. When we need to distinguish among clients we make the dependency on the identity explicit, e.g., $(\mathsf{pk}_{\mathsf{id}}, \mathsf{sk}_{\mathsf{id}})$.*

$\mathsf{MKHS.Sign}(\mathsf{sk}, \Delta, \ell, \mathsf{m}) \to \sigma$. *This algorithm takes as input a secret key sk, a data set identifier Δ (e.g., a string), a label $\ell = (\mathsf{id}, \tau)$ for the message m. It outputs*

$$\sigma \leftarrow \mathsf{HS.Sign}(\mathsf{sk}_{\mathsf{id}}, \Delta, \tau, \mathsf{m}). \tag{1}$$

Without loss of generality we assume that σ includes m.

$\mathsf{MKHS.Eval}(\mathcal{P}, \Delta, \{(\boldsymbol{\sigma}_i, \mathsf{pk}_{\mathsf{id}_i})\}_{i\in[t]}) \to \hat{\boldsymbol{\sigma}}$. *Let $\mathcal{P} = (C, (\ell_1, ..., \ell_{\mathsf{n}}))$, where $C = (\mathsf{n}, 1, \mathsf{q}, \mathsf{L}, \mathsf{R}, \mathsf{G})$ and the $\mathsf{n} \geq t$ labels are of the form $\ell_j = (\mathsf{id}_i, \tau_j)$ for some $i \in [t]$ and $\tau_j \in \mathcal{T}$, where $t \leq \mathsf{T}$.*

The case $t = 1$ *In this case all the n signatures belong to the same user, that is to say, there exists an identity $\mathsf{id} \in \mathsf{ID}$ such that for all $j \in [\mathsf{n}]$ the labels are of the form $\ell = (\mathsf{id}, \tau_j)$ for some $\tau_j \in \mathcal{T}$. Thus, it is possible to run the classical evaluation algorithm of HS and the output of the multi-key evaluation algorithm for $t = 1$ is:*

$$\hat{\boldsymbol{\sigma}} = \hat{\sigma}_{\mathsf{id}} \leftarrow \mathsf{HS.Eval}\big(E_0, \mathsf{pk}_{\mathsf{id}}, (\sigma_1^{\mathsf{id}}, ..., \sigma_{\mathsf{n}}^{\mathsf{id}})\big). \tag{2}$$

The case $t \geq 2$ *In this case the inputs to the labeled program belong to t distinct users. Without loss of generality, we assume that the labels are ordered per client identity, i.e., all the labels between ℓ_{t_j} and $\ell_{\mathsf{t}_{j+1}-1}$ are of the form $(\mathsf{id}_j, *)$. For each $i \in [t]$ the signature vector $\boldsymbol{\sigma}_i$ is $\boldsymbol{\sigma}_i = (\sigma_1^i, ..., \sigma_{\mathsf{n}_i}^i)$ for opportune values $\mathsf{n}_i \in [\mathsf{n} - t + 1]$ satisfying $\sum_{i=1}^{t} \mathsf{n}_i = \mathsf{n}$. Let $\mathsf{t}_i = (\sum_{j=0}^{i-1} \mathsf{n}_j) + 1$, where we set $\mathsf{n}_0 = 0$, then t_i corresponds to the index of first input of identity id_i. The multi-key homomorphic evaluation performs the following $t + 1$ steps.*

Step 0. *Given $\mathcal{P} = (C, (\ell_1, ..., \ell_{\mathsf{n}}))$ retrieve the messages corresponding to the labels $\ell_1, ..., \ell_{\mathsf{n}}$. For notation sake let m_j be the message corresponding to label ℓ_j. Compute the value $\mathsf{y} = C(\mathsf{m}_1, ..., \mathsf{m}_{\mathsf{n}})$. Define a single-input single-output circuit $EQ^{\mathsf{y}}(x)$ that outputs 1 if and only if $x = \mathsf{y}$.*[5] *Construct $E_0 = C \triangleright EQ^{\mathsf{y}} = (\mathsf{n}, 1, \mathsf{q}_0, \mathsf{L}_0, \mathsf{R}_0, \mathsf{G}_0)$. The properties of EQ^{y} imply that:*

$$E_0(x_1, ..., x_{\mathsf{n}}) = 1 \text{ iff } C(x_1, ..., x_{\mathsf{n}}) = \mathsf{y}\ . \tag{3}$$

Note that E_0 can be constructed directly from C and y, moreover

$$E_0(\mathsf{m}_1, ..., \mathsf{m}_{\mathsf{n}}) = 1. \tag{4}$$

Step 1. *We build a n_1-input circuit E_1 that corresponds to a partial evaluation of E_0 on the inputs*

[4] If HS works without these a-priori bounds, it is enough to run $\mathsf{HS.Setup}(1^\lambda)$.
[5] An explicit construction of the circuit EQ^{y} is given in Definition 1 in the Section 2.1.

of identities id_j *with* $j > 1$. *Given* $\mathcal{E}_0 = (E_0, (\ell_1, \dots, \ell_{\mathsf{n}}))$, *the signatures* $\boldsymbol{\sigma_1} = (\sigma_1^1, \dots, \sigma_{\mathsf{n}_1}^1)$ *and the messages* $\mathsf{m}_{\mathsf{n}_1+1}, \dots, \mathsf{m}_{\mathsf{n}}$ *do:*

• *Define the mask circuit* $M_1 = (\mathsf{n}_1, \mathsf{n}, \mathsf{n}, \mathsf{L}_1', \mathsf{R}_1', \mathsf{G}_1')$ *where*

$$\mathsf{L}_1'(j) = \mathsf{R}_1'(j) = \begin{cases} 1 & \text{for } j \in [\mathsf{n}_1] \\ 0 & \text{for } j \in [\mathsf{n}] \setminus [\mathsf{n}_1] \end{cases} \text{ and } \mathsf{G}_1' = \begin{cases} 0 & \text{for } j \in [\mathsf{n}_1] \\ \mathsf{m}_j & \text{for } j \in [\mathsf{n}] \setminus [\mathsf{n}_1] \end{cases}.$$

By construction $M_1(b_1, \dots, , b_{\mathsf{n}_1}) = (b_1, \dots b_{\mathsf{n}_1}, \mathsf{m}_{\mathsf{n}_1+1}, \dots, \mathsf{m}_{\mathsf{n}})$.

• *Compose* M_1 *with* E_0 *to obtain* $E_1 = M_1 \triangleright E_0 = (\mathsf{n}_1, 1, \mathsf{q}_1, \mathsf{L}_1, \mathsf{R}_1, \mathsf{G}_1)$ *where:* $\mathsf{q}_1 = \mathsf{q}_0 + \mathsf{n}$*;* $\mathsf{G}_1 = (\mathsf{G}_1' || \mathsf{G}_0)$*;* $\mathsf{L}_1(g) = \mathsf{L}_1'(g)$ *for* $g \in [\mathsf{n}]$, $\mathsf{L}_1(g) = (\mathsf{L}_0(g - \mathsf{n} + 1) + 1)$ *for* $g \in [\mathsf{n}+1, \mathsf{n}+\mathsf{q}_0]$ *if* $\mathsf{L}_0(g - \mathsf{n} + 1) \neq 0$ *and* 0 *whenever* $\mathsf{L}_0(g - \mathsf{n} + 1) = 0$. *The function* $\mathsf{R}_1(g)$ *is defined analogously. Equation* (4) *implies*

$$E_1(\mathsf{m}_1, \dots, \mathsf{m}_{\mathsf{n}_1}) = 1. \tag{5}$$

• *Compute* $\hat{\sigma}_1 \leftarrow \mathsf{HS.Eval}(E_1, \mathsf{pk}_{\mathsf{id}_1}, \boldsymbol{\sigma}_1)$. *This is possible since* E_1 *is a circuit involving only inputs of client* id_1.

Remark 2. Let $\mathcal{E}_1 = (E_1, (\tau_1, \dots, \tau_{\mathsf{n}_1}))$. *Equation* (5) *and the correctness of the* HS *scheme imply* $\mathsf{HS.Verify}(\mathcal{E}_1, \Delta, \mathsf{pk}_{\mathsf{id}_1}, \hat{\sigma}_1, 1) = 1$.

Step *i* ***for*** $i \in [2, t]$**.** *The goal is to construct an* n_i*-input circuit* E_i *using* $\mathcal{E}_{i-1} = (E_{i-1}, (\tau_{\mathsf{t}_i}, \dots, \tau_{\mathsf{t}_{i+1}-1}))$, Δ, $\mathsf{pk}_{\mathsf{id}_i}$ *and* $\boldsymbol{\sigma_i} = (\sigma_1^i, \dots \sigma_{\mathsf{n}_i}^i)$. *This will be possible using the circuits* $\mathsf{HSV}_i = (\mathsf{n}_{\mathsf{HSV}i}, 1, \mathsf{q}_{\mathsf{HSV}_i}, \mathsf{L}_{\mathsf{HSV}_i}, \mathsf{R}_{\mathsf{HSV}_i}, \mathsf{G}_{\mathsf{HSV}_i})$ *for the (single-key) homomorphic signature verification against the value 1* .[6] *Let* $S_{i-1} = (\mathcal{E}_{i-1}, \Delta, \mathsf{pk}_{\mathsf{id}_{i-1}}, \boldsymbol{\sigma}_{i-1})$ *be a string of* $\mathsf{n}_{\mathsf{HSV}i} = size(S_{i-1})$ *bits. Set* $g_1 = 1$. *The gates of* E_{i-1} *that embed the* n_i *values input by identity* id_i *are located in the interval* $I_i = [g_i, g_i + \mathsf{n}_i]$, *where* $g_i = 3\lg(N_{i-1}) + 2\mathsf{q}_{i-1}\lg(\mathsf{w}_{i-1}) + g_{i-1} + \mathsf{n}_{i-1}$ *(see [18] for an explanation).*

• *Define the mask circuit* $M_i = (\mathsf{n}_i, \mathsf{n}_{\mathsf{HSV}i}, \mathsf{n}_{\mathsf{HSV}i}, \mathsf{L}_i', \mathsf{R}_i', \mathsf{G}_i')$ *where*

$$\mathsf{L}_i'(g) = \mathsf{R}_i'(g) = \begin{cases} 0 & \text{if } g \in [\mathsf{n}_{\mathsf{HSV}i}] \setminus I_i \\ 1 & \text{if } g \in I_i \end{cases} \text{ and } \mathsf{G}_i'(g) = \begin{cases} S_{i-1}(g) & \text{if } g \in [\mathsf{n}_{\mathsf{HSV}3}] \setminus I_i \\ 0 & \text{if } g \in I_i \end{cases}$$

Note that for gates g *in the interval* I_i, $\mathsf{L}_i'(g) = 1$ *and* $\mathsf{G}_i'(g) = 0$ *which means that* M_i *outputs its* n_i *input bits exactly the interval* I_i, *while outside* I_i *the output of* M_i *is constant. In particular:* $M_i(\mathsf{m}_{\mathsf{t}_i}, \dots, \mathsf{m}_{\mathsf{t}_i+\mathsf{n}_i}) = S_{i-1}$.

• *Compose* M_i *with* HSV_i *to obtain* $E_i = M_i \triangleright \mathsf{HSV}_i = (\mathsf{n}_i, 1, \mathsf{q}_i, \mathsf{L}_i, \mathsf{R}_i, \mathsf{G}_i)$ *where:* $\mathsf{q}_i = \mathsf{n}_{\mathsf{HSV}i} + \mathsf{q}_{\mathsf{HSV}_i}$*;* $\mathsf{G}_i = (\mathsf{G}_i' || \mathsf{G}_{\mathsf{HSV}_i})$*;* $\mathsf{L}_i(g) = \mathsf{L}_i'(g)$ *for* $g \in [\mathsf{n}_{\mathsf{HSV}i}]$, $\mathsf{L}_i(g) = \mathsf{L}_{\mathsf{HSV}_i}(g - \mathsf{n}_{\mathsf{HSV}i} + 1) + \mathsf{n}_i$ *for* $g \in [\mathsf{n}_{\mathsf{HSV}i}+1, \mathsf{q}_i]$ *if* $\mathsf{L}_{\mathsf{HSV}_i}(g - \mathsf{n}_{\mathsf{HSV}i}+1) \neq 0$, *and* 0 *otherwise; and* R_i *is defined analogously. . Circuit composition ensures that*[7] $E_i(\mathsf{m}_{\mathsf{t}_i}, \dots, \mathsf{m}_{\mathsf{t}_i+\mathsf{n}_i}) = \mathsf{HS.Verify}(\mathcal{E}_{i-1}, \Delta, \mathsf{pk}_{\mathsf{id}_{i-1}}, \hat{\sigma}_{i-1}, 1)$. *In particular, applying Remark 2 inductively we get:*

$$E_i(\mathsf{m}_{\mathsf{t}_i}, \dots, \mathsf{m}_{\mathsf{t}_i+\mathsf{n}_i}) = 1 \tag{6}$$

[6] The readers can consider the circuit HSV_i to be the representation of $\mathsf{HS.Verify}(\mathcal{E}_{i-1}, \cdot, \cdot, 1)$ where $\mathcal{E}_{i-1}$ is a labelled program for a circuit of size at most $O((\mathsf{n}_{\mathsf{HSV}i-1} + \mathsf{q}_{\mathsf{HSV}_{i-1}})\lg(\mathsf{w}_{\mathsf{HSV}_{i-1}}))$.

[7] With abuse of notation one can think that $E_i(\mathsf{m}_{\mathsf{t}_i}, \dots, \mathsf{m}_{\mathsf{t}_i+\mathsf{n}_i}) = M_i(\mathsf{m}_{\mathsf{t}_i}, \dots, \mathsf{m}_{\mathsf{t}_i+\mathsf{n}_i}) \triangleright \mathsf{HSV}_i = \mathsf{HSV}_i(M_i(\mathsf{m}_{\mathsf{t}_i}, \dots, \mathsf{m}_{\mathsf{t}_i+\mathsf{n}_i}))$. Since $M_i(\mathsf{m}_{\mathsf{t}_i}, \dots, \mathsf{m}_{\mathsf{t}_i+\mathsf{n}_i}) = S_{i-1}$ the claim follows by the definition of HSV_i.

Note that E_i can be constructed directly from $\mathcal{E}_0$ given the values $\mathsf{m}_{\mathsf{t}_i}, \dots, \mathsf{m}_{\mathsf{n}}$ and the public data Δ, $\mathsf{pk}_{\mathsf{id}_j}, \hat{\sigma}_j$ for $j \in [i-1]$. In more details, for $i \in [2,t]$ consider the set of bit strings: $\mathsf{head}_i = (\mathsf{n}_i, 1, \mathsf{q}_i, \mathsf{L}_i, \mathsf{R}_i)$ and $\mathsf{tail}_i = (\tau_{\mathsf{t}_i}, \dots, \tau_{\mathsf{t}_i+\mathsf{n}_i}, \Delta, \mathsf{pk}_{\mathsf{id}_{i-1}}, \hat{\sigma}_{i-1}, \mathsf{G}_{\mathsf{HSV}_i})$. For every $i \in [2,t]$ head_i and tail_i are completely determined by the tags for identity id_{i-1}, the public key $\mathsf{pk}_{\mathsf{id}_{i-1}}$ and the evaluated signature $\hat{\sigma}_{i-1}$. It is immediate to see that head_i and tail_i are respectively the head and the tail of the circuit description of E_i. The heart of the string E_i is where "all the magic" happens:

$$\mathsf{body}_i = (\mathsf{head}_{i-1}, \dots, \mathsf{head}_2, \underbrace{0, \dots, 0}_{(\mathsf{t}_{i+1}-1)=\sum_{j=1}^{i}\mathsf{n}_j} \mathsf{m}_{\mathsf{t}_i}, \dots, \mathsf{m}_{\mathsf{n}}, \mathsf{G}_0, \mathsf{tail}_2, \dots, \mathsf{tail}_i) \tag{7}$$

In particular, for $i = t$ we have:

$$\begin{aligned} E_t &= \Big(\mathsf{head}_t \qquad\qquad \mathsf{body}_t \qquad\qquad \mathsf{tail}_t \Big) \\ &= \Big(\mathsf{head}_t, (\mathsf{head}_{t-1}, \dots, \mathsf{head}_2, \underbrace{0, \dots, 0}_{\mathsf{n}}, \mathsf{G}_0, \mathsf{tail}_2, \dots, \mathsf{tail}_{t-1}), \mathsf{tail}_t\Big) \end{aligned} \tag{8}$$

Equation (8) shows that the circuit E_t is completely determined by the labeled program $\mathcal{E}_0$ (to get the tags and the gate description G_0), the dataset identifier Δ, the public keys $\mathsf{pk}_{\mathsf{id}_i}$ and the signatures $\hat{\sigma}_i$ for $i \in [t]$.

- *Compute $\hat{\sigma}_i \leftarrow \mathsf{HS.Eval}(E_i, \mathsf{pk}_{\mathsf{id}_i}, \boldsymbol{\sigma}_i)$.*

Remark 3. This is possible since E_i is a n_i-input circuit with inputs from the user id_i only. Equation (6) and the correctness of the HS scheme imply that

$$\mathsf{HS.Verify}(\mathcal{E}_i, \Delta, \mathsf{pk}_i, 1, \hat{\sigma}_i) = 1. \tag{9}$$

The output of the multi-key evaluation algorithm is the vector of t signatures: $\hat{\boldsymbol{\sigma}} = (\hat{\sigma}_1, \dots, \hat{\sigma}_t)$.

$\mathsf{MKHS.Verify}(\mathcal{P}, \Delta, \{\mathsf{pk}_{\mathsf{id}}\}_{\mathsf{id}\in\mathcal{P}}, \mathsf{y}, \hat{\boldsymbol{\sigma}}) \to \{0,1\}$. *The verification algorithm parses the labeled program as $\mathcal{P} = (C, (\ell_1 \dots, \ell_{\mathsf{n}}))$ and checks the number $1 \le t \le \mathsf{T}$ of distinct identities present among the n labels.*

The case $t = 1$ *In this case all the inputs to the labeled program $\mathcal{P}$ come from the same user and $\hat{\boldsymbol{\sigma}} = \hat{\sigma}_{\mathsf{id}}$. In other words, all the labels are of the form $\ell_j = (\mathsf{id}, \tau_j)$ for an $\mathsf{id} \in \mathsf{ID}$ and some $\tau_j \in \mathcal{T}$. Set $\mathcal{E}_0 = (C, (\tau_1, \dots, \tau_{\mathsf{n}}))$, notice that we removed the identity from the labels. The multi-key verification algorithm returns the output of*

$$\mathsf{HS.Verify}(\mathcal{E}_0, \Delta, \mathsf{pk}_{\mathsf{id}}, 1, \hat{\sigma}_{\mathsf{id}}). \tag{10}$$

The case $t \ge 2$ *In this case the labeled program $\mathcal{P}$ contains labels with $t \ge 2$ distinct identities and $\hat{\boldsymbol{\sigma}} = (\hat{\sigma}_1, \dots, \hat{\sigma}_t)$. Without loss of generality, we assume that the labels are ordered per client identity and $\mathsf{n}_i \in [\mathsf{n} - t + 1]$ is the number of labels with identity id_i.*
Define $E_0 = (\mathsf{n}, 1, \mathsf{q}_0, \mathsf{L}_0\mathsf{R}_0, \mathsf{G}_0)$ as the circuit $E_0 = C \triangleright EQ^{\mathsf{y}}$, where $EQ^{\mathsf{y}}(x)$ is the a single-input single-output circuit that outputs 1 if and only if $x = \mathsf{y}$. Thus, $E_0(x_1, \dots, x_{\mathsf{n}}) = 1$ whenever $C(x_1, \dots, x_{\mathsf{n}}) = \mathsf{y}$. As noted in the Step 0 of the multi-key homomorphicevaluation algorithm, E_0 is completely determined by $\mathcal{P}$ and y.

To verify the signature $\hat{\boldsymbol{\sigma}}$, the multi-key verification algorithm inductively creates the following strings for $i \in [2, t]$:

$$\mathsf{head}_i = (\mathsf{n}_i, 1, \mathsf{q}_i = \mathsf{n}_{\mathsf{HSV}\,i} + \mathsf{q}_{\mathsf{HSV}_i}, \mathsf{L}_i = (\underbrace{0, \dots, 0}_{(\sum_{j=1}^{i-1} \mathsf{n}_j)-bits}, \overbrace{1, \dots, 1}^{\mathsf{n}_i - bits}, \underbrace{0, \dots, 0}_{(\mathsf{n}-\sum_{j=1}^{i} \mathsf{n}_j)-bits}), \mathsf{R}_i = \mathsf{L}_i)$$

$$\mathsf{tail}_i = (\tau_{\mathsf{t}_{i-1}}, \dots, \tau_{\mathsf{t}_{i-1}+\mathsf{n}_{i-1}}, \Delta, \mathsf{pk}_{\mathsf{id}_{i-1}}, \hat{\sigma}_{i-1}, \mathsf{G}_{\mathsf{HSV}_i})$$

where, the circuit HSV_i is the same as the one explained in MKHS.Eval*, i.e., the HSV_i is the (single-key) homomorphic signature verification against the value 1. At this point the verifier can combine all the pieces to (re)-construct the description of the circuit E_t:*

$$E_t = (\mathsf{head}_t, \dots, \mathsf{head}_2, \underbrace{0, \dots, 0}_{\mathsf{n}}, \mathsf{G}_0, \mathsf{tail}_2, \dots, \mathsf{tail}_t). \tag{11}$$

Let $\mathscr{E}_t = (E_t, (\tau_{\mathsf{t}_t}, \dots, \tau_{\mathsf{n}}))$, where we removed id_t from the labels. The verification returns:

$$\mathsf{HS.Verify}(\mathscr{E}_t, \Delta, \mathsf{pk}_{\mathsf{id}_t}, \hat{\sigma}_t, 1). \tag{12}$$

Remark 4. Note that the E_t constructed by the verifier via Equation (11) coincides with the one created by the evaluator via Equation (8).

3.3 Computing the index where the messages of signer i are embedded

Here we explain the reasoning behind the definition of the index

$$g_i = 3\lg(N_{i-1}) + 2\mathsf{q}_{i-1}\lg(\mathsf{w}_{i-1}) + g_{i-1} + \mathsf{n}_{i-1}$$

in our compiler (step i in Definition 4). Recall that in this step we hold the circuit E_{i-1} and look for the positions in its gate description where the n_i values input by identity id_i are located. This will be an interval $I_i = [g_i, g_i + \mathsf{n}_i]$, for some index g_i. Essentially, g_i should jump over the description of the first part of the circuit $E_{i-1} = (\mathsf{n}_{i-1}, 1, \mathsf{q}_{i-1}, \mathsf{L}_{i-1}, \mathsf{R}_{i-1}, \mathsf{G}_{i-1})$ to select the bits in G_{i-1} that contain the values $\mathsf{m}_{(\mathsf{n}_{i-1}+1)}, \dots, \mathsf{m}_{\mathsf{n}_i}$. The description of the values $\mathsf{n}_{i-1}, 1, \mathsf{q}_{i-1}$ covers the first $3\lg(N_{i-1})$ bits. Then, the left/right input functions $\mathsf{L}_{i-1}, \mathsf{R}_{i-1}$ are two strings of q_{i-1} wires covering additional $2\mathsf{q}_{i-1}\lg(\mathsf{w}_{i-1})$ bits. At this point we enter the gate description of E_{i-1} that brings with an accumulative addend of $g_{i-1} + \mathsf{n}_{i-1} + 1$ bits to reach the position where G_{i-1} contains the first message input by client id_i. By construction $\mathsf{G}_{i-1} = (\mathsf{G}'_{i-1} || \mathsf{G}_{\mathsf{HSV}_{i-1}})$, for consistency let $\mathsf{HSV}_1 = \mathsf{G}_0$. For $i \geq 2$ the gates in G'_{i-1} *embed* (a minor modification of) the string $S_{i-2} = ((E_{i-2}, \ell_{\mathsf{t}_{i-1}}, \dots, \ell_{\mathsf{t}_i - 1}), \mathsf{pk}_{\mathsf{id}_{i-2}}, \boldsymbol{\sigma}_{i-2})$, for consistency let $S_0 = (0..0\mathsf{m}_{\mathsf{n}_1+1}..\mathsf{m}_{\mathsf{n}})$ where the first n_1 entries are 0. When $i = 2$, $\mathsf{G}'_1 = S_0$, therefore the bit that contains the first input by client id_2 is the $\mathsf{t}_2 = (\mathsf{n}_1 + 1)$-th bit in G_1, which is consistent with the formula $g_1 + \mathsf{n}_1$ since we set $g_1 = 1$. Now for $i = 3$, G'_2 equals $S_1 = ((E_1, \ell_{\mathsf{n}_1+1}, \dots, \ell_{\mathsf{t}_2-1}), \mathsf{pk}_{\mathsf{id}_1}, \boldsymbol{\sigma}_1)$, except for the (n_3) bits where G'_1 has the gates initiated to the values input by id_3. This interval begins at the $3\lg(N_1)+2\mathsf{q}_1\lg(\mathsf{w}_1)+\mathsf{n}_1+1 = g_2$ bit of S_1. Therefore $g_3 = 3\lg(N_2) + 2\mathsf{q}_2\lg(\mathsf{w}_2) + g_2 + \mathsf{n}_2$. This explains the recursive definition of g_i.

3.4 Correctness and Succinctness of Matrioska

In what follows we show that the Matrioska scheme satisfies the properties stated in Theorem 2.

Succinctness. By construction, for a computation involving messages from t users, our signatures consist of t signatures of the single-input scheme. It is straightforward to see that if HS signatures have length bounded by some polynomial l, the size of Matrioska's signatures is $\leq t \cdot l$, which is, asymptotically, the same level of succinctness as the MK-HS construction by Fiore *et al.* [17].

Correctness. The following two lemmas reduce the authentication and evaluation correctness of Matrioska multi-key homomorphic signatures to the authentication and evaluation correctness, respectively, of the underlying single-key HS scheme.

Lemma 1. *Let* HS *be a single-key homomorphic signature scheme with authentication correctness, then the multi-key homomorphic signature scheme* $\mathsf{MKHS}(\mathsf{HS},\mathsf{T})$ *obtained from the* Matrioska *compiler of Definition 4 achieves authentication correctness.*

Proof. The definition of authentication correctness for multi-key homomorphic signature schemes requires that the signature σ output by $\mathsf{MKHS.Sign}(\mathsf{sk},\ell,\mathsf{m})$ verifies correctly for the message m as the output of the identity program $\mathcal{I}_\ell = (C_{\mathsf{id}},\ell)$. In details, $C_{\mathsf{id}} : \mathcal{M} \to \mathcal{M}$ is the identity circuit defined as

$$C_{\mathsf{id}} = (\mathsf{n}_{\mathsf{id}} = 1, \mathsf{u}_{\mathsf{id}} = 1, \mathsf{q}_{\mathsf{id}} = 1, \mathsf{L}_{\mathsf{id}} = 1, \mathsf{R} = 1, \mathsf{G} = 0)$$

and $\ell = (\mathsf{id},\tau)$ for some $\tau \in \mathcal{T}$. Formally, we want to show that

$$\mathsf{MKHS.Verify}\big(\mathcal{I}_\ell, \mathsf{pk}_{\mathsf{id}}, \mathsf{m}, \mathsf{MKHS.Sign}(\mathsf{sk}_{\mathsf{id}},\ell,\mathsf{m})\big) = 1.$$

The proof is quite intuitive and reduces to the correctness of the homomorphic signature scheme HS. Let $\sigma \leftarrow \mathsf{MKHS.Sign}(\mathsf{sk},\ell = (\mathsf{id},\tau),\mathsf{m})$. By construction (Equation (1)) σ is the output of $\sigma \leftarrow \mathsf{HS.Sign}(\mathsf{sk}_{\mathsf{id}},\tau,\mathsf{m})$. Since $t = \mathsf{n} = 1$ the multi-key verification algorithm runs the single-key verification circuit on the labelled program $\mathcal{E}_0 = (C,\tau)$, the key $\mathsf{pk}_{\mathsf{id}}$, the value 1 and the signature $\hat{\boldsymbol{\sigma}} = \sigma$ (Equation 10). Thus,

$$\mathsf{MKHS.Verify}(\mathcal{I}_\ell, \mathsf{pk}, \mathsf{m}, \sigma) = \mathsf{HS.Verify}(\mathcal{E}_0, \mathsf{pk}_{\mathsf{id}}, 1, \sigma) = 1$$

by the correctness of the HS scheme.

Lemma 2. *Let* HS *be a single-key homomorphic signature scheme with evaluation correctness, then the multi-key homomorphic signature scheme* $\mathsf{MKHS}(\mathsf{HS},\mathsf{T})$ *obtained from the* Matrioska *compiler of Definition 4 achieves evaluation correctness.*

Proof. The evaluation correctness of Matrioska essentially follows from the evaluation correctness of HS and the way we (inductively) define the circuits E_i. Moreover, notice that our MK-HS scheme is single-hop, therefore we have to prove evaluation correctness with respect to computing on freshly generated signatures (given that authentication correctness is granted by the previous lemma).

Formally, we want to prove that for any labeled program $\mathcal{P} = (C,(\ell_1,\ldots,\ell_{\mathsf{n}}))$ with $C \in \mathcal{F}$ an admissible circuit and $t \leq \mathsf{T}$ distinct keys, if all the n signatures are valid then the signature output by the multi-key evaluation algorithm verifies, that is:

$$\left.\begin{array}{l} \sigma_j \leftarrow \mathsf{MKHS.Sign}(\mathsf{sk}_{\mathsf{id}_j}, \Delta, \ell_j, \mathsf{m}_j), \forall j \in [\mathsf{n}] \\ \\ \hat{\boldsymbol{\sigma}} \leftarrow \mathsf{MKHS.Eval}(\mathcal{P}, \Delta, \{\boldsymbol{\sigma}_i, \mathsf{pk}_{\mathsf{id}_i}\}_{i\in[t]}) \end{array}\right\} \Rightarrow \mathsf{MKHS.Verify}(\mathcal{P}, \Delta, \{\mathsf{pk}_{\mathsf{id}}\}_{\mathsf{id}\in\mathcal{P}}, \mathsf{y}, \hat{\boldsymbol{\sigma}})$$

Given the nature of the Matrioska approach, the proof proceeds by induction.

For $t = 1$, the Matrioska evaluation algorithm returns the output of HS.Eval (see Equation (2)) while the multi-key verification algorithm runs HS.Verify (see Equation (10)). Thus the correctness follows from the evaluation correctness of the single-key homomorphic signature scheme HS.

For $t \geq 2$ it is sufficient to notice that each E_i is defined in such a way that $E_i(\mathsf{m}_{\mathsf{t}_i}, \ldots, \mathsf{m}_{\mathsf{t}_i+\mathsf{n}_i}) = 1$, for $i \in [t]$ (see Equation (6)) and that the labeled program $\mathcal{E}_t$ reconstructed by the verifier coincides with the one defined in the evaluation algorithm. In details, given that the signatures σ_j with $j \in [\mathsf{t}_i, \mathsf{t}_i + \mathsf{n}_i]$ are valid, so is the (single-client) signature $\hat{\sigma}_i \leftarrow \mathsf{HS.Eval}(E_i, \mathsf{pk}_{\mathsf{id}_i}, \boldsymbol{\sigma_i} = (\sigma^i_{\mathsf{t}_i}, \ldots, \sigma^i_{\mathsf{t}_i+\mathsf{n}_i}))$. By the correctness of HS it holds that $\mathsf{HS.Verify}(\mathcal{E}_i, \Delta, \mathsf{pk}_{\mathsf{id}_i}, \hat{\sigma}_i, 1) = 1$ for all $i \in [t]$ where $\mathcal{E}_i = (E_i, (\tau_{\mathsf{t}_i}, \ldots, \tau_{\mathsf{t}_i+t_i}))$. The equality between the t-th labelled program generated during the multi-key homomorphic evaluation and the $\mathcal{E}_t$ constructed in the verification procedure follows from three main observations. First, MKHS.Eval and MKHS.Verify compute the same initial circuit E_0, determined by $\mathscr{P}$ and the value y. Second, the strings head_i and tail_i used in the two algorithms are equivalent. Third, Equations (8) and (11) show that the two algorithms are reconstructing the same labelled program $\mathcal{E}_t$.

Circuit Growth. In what follows we analyze the size growth of the circuits E_i computed by the Matrioska compiler, and use this to prove the bounds in Theorem 2.

Lemma 3. *Let* HS *be a correct single-key homomorphic signature scheme that supports computations on circuits of (maximum) depth d and size s; then the multi-key homomorphic signature scheme* MKHS(HS, T) *obtained from the* Matrioska *compiler of Definition 4 supports homomorphic computations on circuits of size s' and depth d' provided that $s > (s')^{{c_s}^{\mathsf{T}-1}}$ and $d > \max\{d', d_{\mathsf{HSV}}((s')^{{c_s}^{\mathsf{T}-1}}, \lambda)\}$, where d_{HSV} and c_s are a function and a non-negative constant that depend on the single-key scheme* HS.

Proof. For $t = 1$, MKHS is running the plain algorithms of HS. Therefore MKHS supports circuits of size $s' < s$ and depth $d' < \max\{d, d_{\mathsf{HSV}}(s)\}$.

For $t > 1$ the Matrioska compiler runs HS.Eval and HS.Verify on every E_i including E_t. Since $\{E_i\}_{i\in[t]}$ is a sequence of circuits of increasing size and depth we need to make sure that the circuit given as input to MKHS will grow into an E_t that is supported by HS.

We begin by analysing the size growth; this will be useful to obtain the bound on the depth. It is easy to see that $size(E_0) \sim size(E_1)$, since we are only adding a few gates. The actual growth begins with $i = 2$. From this point on, in fact, the verification circuit HSV_i is contained in E_i. We assume that on input a circuit Z the verification circuit HSV has size $size(\mathsf{HSV}) \sim size(Z)^{c_s}$, for an opportune constant value $c_s \geq 1$ (dependent on the HS scheme). This assumption follows simply by the fact that the HS algorithms must run in polynomial time and thus can be represented with polynomial-size circuits. By construction $size(E_i) = size(E_{i-1})^{c_s}$ for every $i \in [2, t]$. Let s' denote the size of the circuit C given in input to MKHS.Eval, then it is clear that it is sufficient to have $s > size(E_t) = (s')^{{c_s}^{t-1}}$, which proves the claim on the size in Lemma 3 and in Theorem 2.

Let d_{HSV} be a function that expresses the depth of the circuit HSV as a function of the length of its inputs. In Matrioska the inputs to HSV are a circuit E_i and a few more elements depending only on the security parameter λ. Since the size of circuits E_i grow in size with i we consider the bound for the largest $size(E_t) = (s')^{{c_s}^{t-1}}$. In this case $depth(\mathsf{HSV}_t) \sim d_{\mathsf{HSV}}(size(E_{t-1}), \lambda) + 1 = d_{\mathsf{HSV}}(s'^{{c_s}^{t-1}}, \lambda)$. Therefore it must hold that d' and $d_{\mathsf{HSV}}(s'^{{c_s}^{t-1}}, \lambda) < d$. Letting d be greater than the maximum between d' and $d_{\mathsf{HSV}}(s'^{{c_s}^{t-1}}, \lambda)$ we ensure that E_i is supported by HS, and thus by

MKHS. It is worth noticing that a polynomial-time HS.Verify implies the depth function $d_{\mathsf{HSV}}(\cdot)$ to be, in the worst case, a polynomial.

In more details, in our representation of circuits the largest factor in the size of a circuit is determined by the description of the functions L and R. In particular, for a circuit C with $\mathsf{q} >> \mathsf{n}$ the asymptotic bound is $size(C) \sim \mathsf{q}\lg(\mathsf{q})$, where we approximate the number of wires w with the number of gates q.[8]

Let q_0 denote the number of gates in E_0, then: $size(E_0) \sim \mathsf{q}_0 \lg(\mathsf{q}_0)$. Let n be the total number of input to the computation (in case $\mathsf{n}{=}t$ each input belongs to a different user) then $size(E_1) \sim (\mathsf{q}_0 + \mathsf{n})\lg(\mathsf{q}_0 + \mathsf{n})$, since $\mathsf{n} < \mathsf{q}_0$ in this analysis we consider

$$size(E_1) \sim 2\mathsf{q}_0 \lg(2\mathsf{q}_0). \tag{13}$$

The actual growth begins with $i = 2$. From this point on, in fact, the verification circuit HSV_i is contained in E_i. Without loss of generality, we assume that on input a circuit of size Z the verification circuit $size(\mathsf{HSV})$ has size $size(\mathsf{HSV}) \sim size(Z)^{c_s}$, for a constant value $c_s \geq 1$ (dependent on the HS scheme). In our compiler this translates to $\mathsf{q}_{\mathsf{HSV}_i} \lg(\mathsf{q}_{\mathsf{HSV}_i}) \sim (\mathsf{q}_{i-1} \lg(\mathsf{q}_{i-1}))^{c_s}$, thus:

$$\begin{aligned} size(E_2) = size(M_2 \triangleright \mathsf{HSV}_2) &= |(1, 1, \mathsf{q}_2, \mathsf{L}_2, \mathsf{R}_2, \mathsf{G}_2)| \\ &\sim \mathsf{q}_2 \lg(\mathsf{q}_2) && (14) \\ &\sim \mathsf{q}_{\mathsf{HSV}_2} \lg(\mathsf{q}_{\mathsf{HSV}_2}) && (15) \\ &\sim (2\mathsf{q}_0 \lg(2\mathsf{q}_0))^{c_s}. && (16) \end{aligned}$$

Where (14) is the usual approximation of the circuit's size with the number of gates, (15) comes from the fact that[9] $\mathsf{q}_2 = \mathsf{n}_{\mathsf{HSV}_2} + \mathsf{q}_{\mathsf{HSV}_2} \sim \mathsf{q}_{\mathsf{HSV}_2}$ and (16) is implied by our assumption $size(\mathsf{HSV}_2) \sim size(E_1)^{c_s}$ and (13). Following this reasoning inductively we get: $size(E_t) \sim (\mathsf{q}_0 \lg(\mathsf{q}_0))^{{c_s}^{t-1}}$. In other words, let s' denote the size of the circuit C given in input to MKHS.Eval, then $size(E_t) = {s'}^{{c_s}^{t-1}} < s$. Regarding the depth growth we notice that if the function $d_{\mathsf{HSV}}(z, \lambda)$ is polynomial there exists a constant $c_d \geq 1$ such that $d_{\mathsf{HSV}}(z, \lambda) = z^{c_d}$. Then $depth(E_t) = size(E_{t-1})^{c_d} = s^{c_d \cdot {c_s}^{1-t}}$. However, if $d_{\mathsf{HSV}}(z, \lambda)$ is logarithmic then $depth(E_t) = \log(size(E_{t-1})) = ({c_s}^{1-t})s$.

3.5 Security of Matrioska

In this section we argue that Matrioska MKHS schemes are unforgeable provided that so is the underlying HS scheme. For the proof we rely on Proposition 1 from [17], which allows for a simpler treating of corruption queries.

Lemma 4. *Let* HS *be a secure single-key homomorphic signature scheme. Then the multi-key homomorphic signature scheme* $\mathsf{MKHS}(\mathsf{HS}, \mathsf{T})$ *obtained from the* Matrioska *compiler of Definition 4 is secure. In particular, for any PPT adversary* $\mathcal{A}$ *making signing queries on at most* $Q_{\mathsf{id}} = \mathsf{poly}(\lambda)$ *distinct identities, there is a PPT algorithm* $\mathcal{B}$ *such that:* $\mathrm{Adv}^{\mathsf{MKHS}}_{\mathcal{A}} \leq Q_{\mathsf{id}} \cdot \mathrm{Adv}^{\mathsf{HS}}_{\mathcal{B}}$.

Before giving the detailed proof we provide the intuition of the reduction flow. A forger against our MKHS scheme must create a forgery for the HS scheme for at least one of the users, say $\mathsf{id}_{i^\star}$, involved in the computation. Thus the reduction $\mathcal{B}$, on input a public key pk, makes a guess for

[8] Approximating $\mathsf{w} = \mathsf{q} + \mathsf{n}$ with q is a quite tight in our case, since all E_i are circuits with one single input.

[9] to upperbound we could put a factor 2 in this estimate: $\mathsf{q}_2 < 2\mathsf{q}_{\mathsf{HSV}_2}$ but since this is an asymptotic estimate and $\mathsf{q}_{\mathsf{HSV}_2} > 2$ it would not change much.

$j^* = i^\star$, programs $\mathsf{pk}_{\mathsf{id}_{j^*}} = \mathsf{pk}$ and generates all the other keys. This allows $\mathcal{B}$ to perfectly simulate all the signing queries (perfectly hiding j^* to $\mathcal{A}$).

When $\mathcal{A}$ returns $(\mathcal{P}^*, \Delta^*, \{\mathsf{pk}^*_{\mathsf{id}}\}_{\mathsf{id}\in\mathcal{P}^*}, \mathsf{y}^*, \sigma^*)$, with $\sigma^* = (\hat{\sigma}^*_1, \ldots, \hat{\sigma}^*_t)$, the crucial part of the proof is showing the existence of an index $i^\star$ such that $\hat{\sigma}^*_{i^\star}$ is a forgery for HS. Specifically:

- σ^* is of type-1 (Δ^* is new). Then $i^\star = t$ and $\hat{\sigma}^*_t$ is a type-1 forgery against HS.
- σ^* is of type-2. This means: $E_0(\mathsf{m}_1, \ldots, \mathsf{m}_{\mathsf{n}}) = 0$ while $\mathsf{HS.Verify}(\mathcal{E}_t, \mathsf{pk}_{\mathsf{id}_t}, 1, \hat{\sigma}^*_t) = 1$. Then we show that there must exist a "forking index" $i^\star \in [t]$ such that $E_{i-1}(\mathsf{m}_{\mathsf{t}_{i-1}}, \ldots, \mathsf{m}_{\mathsf{t}_{i-1}+\mathsf{n}_{i-1}}) = 0$ but $\mathsf{HS.Verify}(\mathcal{E}_i, \mathsf{pk}_{\mathsf{id}_i}, \hat{\sigma}^*_i, 1) = 1$, that is, $\hat{\sigma}^*_{i^\star}$ is a type-2 forgery against HS for the labeled program $\mathcal{E}_i$.
- σ^* is of type-3. If $t = 1$, then $i^\star = 1$ and $\hat{\sigma}^*_1$ is a type-3 forgery against HS. If $t > 1$, let $i \in [t]$ be the first index such that $\exists\, j \in [\mathsf{n}] : \ell_j = (\mathsf{id}_i, \tau_j) \notin L_{\Delta^*}$, *i.e.,* the first identity for which a type-3 forgery condition holds. Then, either $\hat{\sigma}^*_i$ is a type-3 forgery for HS for identity id_i (and thus $i^\star = i$); or there is $i^\star > i$ such that $\hat{\sigma}^*_{i^\star}$ is a type-2 forgery against identity $\mathsf{id}_{i^\star}$. The latter can be argued by showing the existence of a "forking index" as in the previous case. In a nutshell, a type-3 forgery against MKHS comes either from a type-3 forgery at some index i, or, the i-th signature is incorrect and thus there must be a type-2 forgery at a later index to cheat on the fact that verification at index i is correct.

Therefore, if $j^* = i^\star$ (which happens with non-negligible probability $1/Q_{\mathsf{id}}$), $\mathcal{B}$ can convert $\mathcal{A}$'s forgery into one for its challenger. The detailed proof of Lemma 4 is given below.

Proof. We define a reduction $\mathcal{B}$ between an $\mathsf{MK\text{-}HomUF\text{-}CMA}$ forger $\mathcal{A}$ and an $\mathsf{HomUF\text{-}CMA}$ challenger $\mathcal{C}$. The reduction begins by initializing the identity counter q to 1 and by choosing a (random) index $j^* \leftarrow [Q_{\mathsf{id}}]$ as a guess for an identity on which $\mathcal{A}$ will make a forgery.

Setup. In the setup phase $\mathcal{B}$ starts the $\mathsf{HomUF\text{-}CMA}$ game for the scheme HS. The reduction uses the public parameters of the HS scheme given by its $\mathsf{HomUF\text{-}CMA}$ challenger $\mathcal{C}$ to generate pp for the MKHS scheme (*e.g.,* adding the ID set, redefining the labels). $\mathcal{B}$ sends pp to the adversary $\mathcal{A}$, and stores the public key pk provided by $\mathcal{C}$.

Sign queries. In the sign queries, the reduction $\mathcal{B}$ answers to queries $(\Delta, \ell = (\mathsf{id}, \tau), \mathsf{m})$ as follows:

1. If this is the first query for the dataset Δ, the reduction initializes an empty list $L_\Delta = \emptyset$ and proceeds to step 2.
2. If this is the first query with identity id *and* $q \neq j^*$, generate keys for the identity id running $(\mathsf{sk}_q, \mathsf{pk}_q) \leftarrow \mathsf{MKHS.KeyGen}(\mathsf{pp})$. If this is the first query with identity id *and* $q = j^*$, set $\mathsf{pk}_{j^*} = \mathsf{pk}$ ($\mathcal{A}$ is generating the user that $\mathcal{B}$ has guessed as to be the target for the forgery). Update the identity-index map $\omega : \mathsf{ID} \to [Q_{\mathsf{id}}]$ with $\omega(\mathsf{id}) = q$ and increase the identity counter $q \leftarrow q + 1$. Proceed to step 3 and 4.
3. If the query has not been asked before, (*i.e.,* $(\ell, \mathsf{m}) \notin L_\Delta$) *and* $\omega(\mathsf{id}) = i \neq j^*$, compute $\sigma \leftarrow \mathsf{MKHS.Sign}(\mathsf{sk}_i, \ell, \mathsf{m})$ (notice that in this case $\mathcal{B}$ knows the secret key). If $(\ell, \mathsf{m}) \notin L_\Delta$ *and* $\omega(\mathsf{id}) = j^*$, $\mathcal{B}$ queries its challenger $\mathcal{C}$ with $(\Delta, \tau, \mathsf{m})$ to obtain a signature σ. Finally, in both cases, the reduction updates the list of queried messages for the database Δ: $L_\Delta \leftarrow L_\Delta \cup \{(\ell, \mathsf{m})\}$ and returns (σ, pk_i) to $\mathcal{A}$ ($i \in [q]$).
4. If the query has the same label of a previous query on the same dataset, *i.e.,* there exists a message $\mathsf{m}' \in \mathcal{M}$ such that $(\ell, \mathsf{m}') \in L_\Delta$, ignore the query.

It is easy to see that up to this point $\mathcal{B}$ perfectly simulates the $\mathsf{MK\text{-}HomUF\text{-}CMA}$ game to $\mathcal{A}$.

Forgery. Let $(\mathscr{P}^*, \Delta^*, \{\mathsf{pk}^*_{\mathsf{id}}\}_{\mathsf{id}\in\mathscr{P}^*}, \mathsf{y}^*, \sigma^*)$ denote the output of $\mathscr{A}$ at the end of the MK-HomUF-CMA security experiment (simulated by $\mathscr{B}$). Let id^* denote the identity corresponding to the index j^* chosen by $\mathscr{B}$, *i.e.,* $\omega(\mathsf{id}^*) = j^*$. If $\mathsf{id}^* \notin \mathscr{P}^*$ the reduction aborts. In this case indeed, $\mathscr{B}$ has for sure failed to guess one of the identities involved in the forgery made by $\mathscr{A}$. Otherwise, in what follows we show that the forgery $(\mathscr{P}^*, \Delta^*, \{\mathsf{pk}^*_{\mathsf{id}}\}_{\mathsf{id}\in\mathscr{P}^*}, \mathsf{y}^*, \sigma^*)$ can be converted (except with some non-negligible error probability related to the wrong guess of j^*) into a single-key forgery for the HomUF-CMA experiment. In what follows we do an analysis case by case.

Single user. If $\mathscr{P}^*$ is a computation that involves a single user only, the reduction is perfect, *i.e.,* $\mathscr{B}$ can turn every forgery (of any type) output by $\mathscr{A}$ against MKHS into a forgery against HS by removing the identity id^* from the labels. Indeed, notice that if $\mathscr{B}$ reached this point, it did not abort, and thus $\mathsf{id}^* \in \mathscr{P}^*$.

Multi-user programs. If $\mathscr{P}^*$ involves $t > 1$ users, the reduction proceeds as follows.

Type-1 Forgery. Namely, it holds both $\mathsf{MKHS.Verify}(\mathscr{P}^*, \Delta^*, \{\mathsf{pk}^*_{\mathsf{id}}\}_{\mathsf{id}\in\mathscr{P}^*}, \mathsf{y}^*, \sigma^*) = 1$ and $L_{\Delta^*} = \emptyset$. We show that this corresponds to a type-1 forgery against HS for the t-th key in $\mathscr{P}^*$. By construction (see Equation (12)), MKHS.Verify outputs 1 if and only if $\mathsf{HS.Verify}((E_t, (\tau^*_{\mathsf{t}_t}, \dots, \tau^*_{\mathsf{n}})), \Delta^*, \mathsf{pk}^*_{\mathsf{id}_t}, 1, \hat{\sigma}^*_t) = 1$. Moreover, since $L_{\Delta^*} = \emptyset$, the reduction never queried its challenger $\mathcal{C}$ on the dataset Δ^* either. The last two conditions ensure that $(E_t, (\tau^*_{\mathsf{t}_t}, \dots, \tau^*_{\mathsf{n}})), \Delta^*, \mathsf{pk}^*_{\mathsf{id}_t}, 1, \hat{\sigma}^*_t)$ is a type-1 forgery against HS for the key pair $(\mathsf{pk}_{\mathsf{id}_t}, \mathsf{sk}_{\mathsf{id}_t})$.

Therefore, in this case the reduction $\mathscr{B}$ returns $(E_t, (\tau^*_{\mathsf{t}_t}, \dots, \tau^*_{\mathsf{n}})), \Delta^*, \mathsf{pk}^*_{\mathsf{id}_t}, 1, \hat{\sigma}^*_t)$ to its challenger, if $j^* = \omega(\mathsf{id}_t)$ (*i.e.,* $\mathsf{pk}_{\mathsf{id}_t} = \mathsf{pk}_{j^*}$), and aborts otherwise.

Remark 5. The adversary $\mathscr{A}$ can possibly produce type-1 forgeries also for identities id_i with $i < t$. In this case, however, L^*_Δ is empty and therefore it is impossible to run the (single-key) verification algorithm on the circuits E_i for $i < t$, as this would require the knowledge of the messages $\mathsf{m}_{\mathsf{t}_i+\mathsf{n}_i+1}, \dots, \mathsf{m}_{\mathsf{n}}$ input by the last $t - i$ users.

Type-2 Forgery. Namely, $\mathscr{A}$ returns $(\mathscr{P}^*, \Delta^*, \{\mathsf{pk}^*_{\mathsf{id}}\}_{\mathsf{id}\in\mathscr{P}^*}, \mathsf{y}^*, \sigma^*)$ such that $\mathsf{MKHS.Verify}(\mathscr{P}^*, \Delta^*, \{\mathsf{pk}^*_{\mathsf{id}}\}_{\mathsf{id}\in\mathscr{P}^*}, \mathsf{y}^*, \sigma^*)=1$ and $\mathsf{y}^* \neq C^*(\mathsf{m}_1, \dots, \mathsf{m}_{\mathsf{n}})$, $\mathsf{m}_1, \dots, \mathsf{m}_{\mathsf{n}}$ are the messages queries by $\mathscr{A}$ for the respective labels in $\mathscr{P}^*$. In the following claim, we formally show that from any type-2 forgery against the MKHS scheme, it is possible to extract a type-2 forgery against the HS scheme (corresponding to at least one of the users involved in $\mathscr{P}^*$).

Claim. Let $t \geq 2$, and let $(\mathscr{P}^*, \Delta^*, \{\mathsf{pk}^*_{\mathsf{id}}\}_{\mathsf{id}\in\mathscr{P}^*}, \mathsf{y}^*, \sigma^*)$ be such that $\mathsf{MKHS.Verify}(\mathscr{P}^*, \Delta^*, \{\mathsf{pk}^*_{\mathsf{id}}\}_{\mathsf{id}\in\mathscr{P}^*}, \mathsf{y}^*, \sigma^*)=1$ and $\mathsf{y}^* \neq C^*(\mathsf{m}_1, \dots, \mathsf{m}_{\mathsf{n}})$, with $\sigma^* = (\hat{\sigma}^*_1, \dots, \hat{\sigma}^*_t)$. Then, there exists (at least) one index $i \in [t]$ such that $\hat{\sigma}^*_i$ is a type-2 forgery against the HS scheme (for an opportune function).

Proof. The claim follows from this inductive reasoning. Consider $\hat{\boldsymbol{\sigma}}^* = (\hat{\sigma}^*_1, \dots, \hat{\sigma}^*_t)$. By definition $E_0(\mathsf{m}_1, \dots, \mathsf{m}_{\mathsf{n}}) = 1$ if and only if $C^*(\mathsf{m}_1, \dots, \mathsf{m}_{\mathsf{n}}) = \mathsf{y}$ (see Equation (3)). Since $(\mathscr{P}^*, \Delta^*, \{\mathsf{pk}^*_{\mathsf{id}}\}_{\mathsf{id}\in\mathscr{P}^*}, \mathsf{y}^*, \sigma^*)$ is a type-2 forgery $\mathsf{y}^* \neq \mathsf{y}$. Therefore $E_0(\mathsf{m}_1, \dots, \mathsf{m}_{\mathsf{n}}) = 0$. The correctness of the Matrioska compiler therefore implies that $E_1(\mathsf{m}_1, \dots, \mathsf{m}_{\mathsf{n}_1}) = 0$ as well, where E_1 is the circuit defined in MKHS.Eval with the messages of identities id_j, with $j > 1$, hardwired.

Given the signature $\hat{\sigma}^*_1$ there are two possible cases: either $\mathsf{HS.Verify}(\mathscr{E}_1, \mathsf{pk}_{\mathsf{id}_1}, \hat{\sigma}^*_1, 1) = 1$ or not. In the first case, $(\mathscr{E}_1, \mathsf{pk}_{\mathsf{id}_1}, \hat{\sigma}^*_1, 1)$ is a type-2 forgery against the HS scheme for the key-pair $(\mathsf{pk}_1, \mathsf{sk}_1)$, and thus we have found our forgery and the claim is proven with index $i = 1$. Otherwise, we proceed inductively to the next identity to show that the claim can be proven for $i > 1$.

By induction, let $i > 1$ and assume $E_{i-1}(\mathsf{m}_{\mathsf{t}_{i-1}}, \ldots, \mathsf{m}_{\mathsf{t}_{i-1}+\mathsf{n}_{i-1}}) = 0$. By construction of Matrioska it holds $E_i(\mathsf{m}_{\mathsf{t}_i}, \ldots, \mathsf{m}_{\mathsf{t}_i+\mathsf{n}_i}) = 0$. Similarly to the case $i = 1$, note that for the signature $\hat{\sigma}_i^*$ there are two possible cases: either $\mathsf{HS.Verify}(\mathscr{E}_i, \mathsf{pk}_{\mathsf{id}_i}, \hat{\sigma}_i^*, 1) = 1$ or not. In the first case $(\mathscr{E}_i, \mathsf{pk}_{\mathsf{id}_i}, \hat{\sigma}_i^*, 1)$ is a type-2 forgery against the HS scheme for the key-pair $(\mathsf{pk}_i, \mathsf{sk}_i)$, and thus we have found our forgery and the claim is proven with this index i. Otherwise, we proceed with index $i + 1$.

Finally, to show that such index i must exist, we notice that we cannot reach $i = t + 1$ without finding a forgery. In fact, for $i = t + 1$ we would have $E_t(\mathsf{m}_{\mathsf{t}_t}, \ldots, \mathsf{m}_{\mathsf{n}}) = 0$. However, by definition of type-2 forgery in MK-HomUF-CMA we have that $\mathsf{MKHS}(\mathscr{P}^*, \Delta^*, \{\mathsf{pk}^*_{\mathsf{id}}\}_{\mathsf{id}\in\mathscr{P}^*}, \mathsf{y}^*, \sigma^*) = 1$, that is $\mathsf{HS.Verify}(\mathscr{E}_t, \mathsf{pk}_{\mathsf{id}_t}, 1, \hat{\sigma}_t^*) = 1$ (see Equation (12)). This immediately shows that $(\mathscr{E}_t, \mathsf{pk}_{\mathsf{id}_t}, 1, \hat{\sigma}_t^*)$ is a type-2 forgery against HS for the key $\mathsf{pk}_{\mathsf{id}_t}$. This completes the proof of the claim.

Given the type-2 forgery produced by $\mathscr{A}$, $\mathscr{B}$ builds the circuit E_{j^*} using $\mathscr{P}^* = (C^*, (\ell_1, \ldots, \ell_{\mathsf{n}}))$, the messages stored in L_{Δ^*} and the signatures $\hat{\sigma}_i^*$ for $i < j^*$. Let i be the index whose existence is granted by the previous claim. If $i = j^*$, $\mathscr{B}$ outputs to its challenger $\mathcal{C}$ the tuple $(\widetilde{\mathscr{E}_{j^*}} = (E_{j^*}, (\tau_{\mathsf{t}_{j^*}}, \ldots, \tau_{\mathsf{t}_{j^*}+\mathsf{n}_{j^*}})), 1, \hat{\sigma}_{j^*}^*)$ as its type-2 forgery against HS for the key pk_{j^*}. Otherwise, the reduction aborts as the guess of j^* was incorrect and $\hat{\sigma}_{j^*}^*$ is not guaranteed to be a forgery.

Type-3 Forgery. Namely, $\mathscr{A}$ returns $(\mathscr{P}^*, \Delta^*, \{\mathsf{pk}^*_{\mathsf{id}}\}_{\mathsf{id}\in\mathscr{P}^*}, \mathsf{y}^*, \sigma^*)$ such that $\mathsf{MKHS.Verify}(\mathscr{P}^*, \Delta^*, \{\mathsf{pk}^*_{\mathsf{id}}\}_{\mathsf{id}\in\mathscr{P}^*}, \mathsf{y}^*, \sigma^*)$=1 and there exists one label, say $\ell_{i^\star} \in \mathscr{P}^*$, for which no sign query was performed, *i.e.,* $(\ell_{i^\star} = (\mathsf{id}_{i^\star}, \tau_{i^\star}), \cdot) \notin L_{\Delta^*}$. In the following claim we show that any such type-3 forgery against MKHS reduces to either a type-3 or a type-2 forgery against HS.

Claim. Let $t \geq 2$. If, at the end of the MK-HomUF-CMA security experiment, $\mathscr{A}$ outputs a type-3 forgery for the label $\ell_{i^\star} = (\mathsf{id}_{i^\star}, \tau_{i^\star}) \in \mathscr{P}^*$ then either (1) the forgery reduces to a type-3 forgery against HS for the identity $\mathsf{id}_{i^\star}$, or (2) there is (at least one) type-2 forgery against HS for an identity $\mathsf{id}_i \in \mathscr{P}^*$ with $i > i^\star$.

Proof. Let $(\mathscr{P}^*, \Delta^*, \{\mathsf{pk}^*_{\mathsf{id}}\}_{\mathsf{id}\in\mathscr{P}^*}, \mathsf{y}^*, \sigma^*)$ be the type-3 forgery output by the adversary at the end of the MK-HomUF-CMA experiment. Since no sign query for the label $\ell_{i^\star}$ was performed during the security experiment, it is impossible to reconstruct any circuit E_i for $i < i^\star$. The motivation is that by construction the description of E_i includes all the messages m_j with $j \geq \mathsf{t}_i$, including $\mathsf{m}_{i^\star}$ (see Equation (7)). Therefore the first circuit that is publicly reconstructible is $E_{i^\star}$. Let $\mathscr{E}_{i^\star} = (E_{i^\star}, (\tau_{\mathsf{t}_{i^\star}}, \ldots, \tau_{\mathsf{t}_{i^\star}+\mathsf{n}_{i^\star}}))$, there are two possible cases: either $\mathsf{HS.Verify}(\mathscr{E}_{i^\star}, \Delta^*, \mathsf{pk}_{i^\star}, 1, \hat{\sigma}_{i^\star}^*) = 1$ or it equals 0. It is immediate to see that in case the verification procedure outputs 1 $(\mathscr{E}_{i^\star}, \Delta^*, \mathsf{pk}_{i^\star}, 1, \hat{\sigma}_{i^\star}^*)$ satisfies all the requirements for a type-3 forgery on the key $\mathsf{pk}_{i^\star}$ against the scheme HS. Otherwise, the verification of the $i^\star$-th circuit fails, *i.e.,* $\mathsf{HS.Verify}(\mathscr{E}_{i^\star}, \Delta^*, \mathsf{pk}_{i^\star}, 1, \hat{\sigma}_{i^\star}^*) = 0$, and the correctness of the Matrioska compiler implies that $E_j(\mathsf{m}_{\mathsf{t}_j}, \ldots, \mathsf{m}_{\mathsf{t}_j+\mathsf{n}_j}) = 0$ for all $j > i^\star$. However, by definition of type-3 forgery the final (multi-key) verification outputs 1. We are now in a situation similar to the one of type=2 forgeries. In particular for MKHS.Verify to output 1, it must hold that $\mathsf{HS.Verify}(\mathscr{E}_t, \Delta^*, \mathsf{pk}_{\mathsf{id}_t}, \hat{\sigma}_t^*, 1) = 1$ (see Equation (12)). Therefore there must be a type-2 forgery against the HS scheme for one identity id_j with $i^\star < j \leq t$. The latter follows by the same argument used in the previous Claim (details omitted). This concludes the proof of the claim.

In light of the claim above, from a type-3 forgery $(\mathscr{P}^*, \Delta^*, \{\mathsf{pk}^*_{\mathsf{id}}\}_{\mathsf{id}\in\mathscr{P}^*}, \mathsf{y}^*, \sigma^*)$ the reduction $\mathscr{B}$ can derive either a type-3 forgery against its HomUF-CMA challenger (if $\mathsf{id}_{i^\star} = \mathsf{id}_{j^*}$ and $\mathsf{HS.Verify}(\mathscr{E}_{j^*}, \Delta^*, \mathsf{pk}_{j^*}, 1, \hat{\sigma}_{j^*}^*) = 1$) or a type-2 forgery (if $j^* > i^\star$ and $\mathsf{HS.Verify}(\mathscr{E}_{j^*}, \Delta^*, \mathsf{pk}_{j^*}, 1, \hat{\sigma}_{j^*}^*) = 1$ while

$E_{j^*}(\mathsf{m}_{\mathsf{t}_{j^*}}, \ldots, \mathsf{m}_{\mathsf{t}_{j^*}+\mathsf{n}_{j^*}}) = 0$, note that since $j^* > i^\star$ we know all the messages needed to define E_{j^*}).

Putting together all the cases analyzed above, one can see that, when $\mathcal{B}$ does not abort, it provides a perfect simulation to $\mathcal{A}$ and always finds a forgery against HS. Hence, $\mathrm{Adv}^{\mathsf{HS}}_{\mathcal{B}} = \mathrm{Adv}^{\mathsf{MKHS}}_{\mathcal{A}} \Pr[\mathcal{B} \text{ does not abort}]$. Since the simulation provided by $\mathcal{B}$ to $\mathcal{A}$ is perfect, the index j^* is completely hidden to $\mathcal{A}$. Also, $\mathcal{B}$ does not abort when j^* equals an appropriate index (in each forgery case), which happens with probability at least $1/Q_{\mathsf{id}}$.

4 Conclusions and Future work

In this paper, we presented Matrioska the first generic compiler based on falsifiable assumptions that establishes a formal connection between single-key HS and multi-key HS schemes. Matrioska introduces an original mechanism to gain multi-key features by levering the homomorphic property of a single-key HS scheme. The resulting signatures are succinct in the sense that their length depends solely on the number of signers involved in the homomorphic computation, and not on the total number of signatures input. Unfortunately, constructions obtained with Matrioska are of limited efficiency, as they require the single-key HS scheme to support circuits of size exponentially large in the maximum number of distinct signers involved in the computation. Achieving full signature succinctness remains an interesting goal for further developments, as well as investigating if Matrioska's approach could be used to enhance other cryptographic primitives with multi-key features.

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A Details of the three-client three-input case for Matrioska

Consider the case in which we want to authenticate the result of a *three*-input circuit $C = (3, 1, \mathsf{q}_C, \mathsf{L}_C, \mathsf{R}_C, \mathsf{G}_C)$ evaluated on *three* messages, each signed by a distinct client. For notation sake, we assume the clients have identities $\mathsf{id}_1 = 1, \mathsf{id}_2 = 2$ and $\mathsf{id}_3 = 3$. Let $\mathsf{m}_i \in \{0, 1\}$ denote the input of party $i \in [3]$, and $\sigma_i \leftarrow \mathsf{HS.Sign}(\mathsf{sk}_i, \ell_i, \mathsf{m}_i)$ be the corresponding signature. Note that each σ_1, σ_2 and σ_3 is generated using a different secret key of the HS scheme. Moreover, in this example we are deliberately removing dataset identifiers for ease of exposition.

Step 0. Given the labeled program $\mathscr{P} = (C, (\ell_1, \ell_2, \ell_3))$ and the three messages $\mathsf{m}_1, \mathsf{m}_2, \mathsf{m}_3$, compute

the value $\mathsf{y} = C(\mathsf{m}_1, \mathsf{m}_2, \mathsf{m}_3)$. Let $E_0 = (3, 1, \mathsf{q}_0, \mathsf{L}_0, \mathsf{R}_0, \mathsf{G}_0)$ be a circuit satisfying $E_0(x_1, x_2, x_3) = 1$ iff $C(x_1, x_2, x_3) = \mathsf{y}$, *e.g.,* $E_0 = C \triangleright EQ^{\mathsf{y}}$. Note that E_0 can be constructed using C and y solely without, knowing the values $\mathsf{m}_1, \mathsf{m}_2, \mathsf{m}_3$ (see the definition given in the Section 2.1). By construction it holds that:

$$E_0(\mathsf{m}_1, \mathsf{m}_2, \mathsf{m}_3) = 1. \tag{17}$$

Step 1. We build a single-input circuit E_1 that corresponds to E_0 where the last two inputs m_2 and m_3 are fixed and hardwired into it. In this way, we obtain a single-input single-client circuit on which we can run $\mathsf{HS.Eval}$ using the public key pk_1. In more details, given $\mathcal{E}_0 = (E_0, (\ell_1, \ell_2, \ell_3))$, the signature σ_1 and the messages $\mathsf{m}_2, \mathsf{m}_3$:

• Define a *mask* circuit $M_1 = (1, 3, 3, \mathsf{L}'_1, \mathsf{R}'_1, \mathsf{G}'_1)$ where $\mathsf{L}'_1 = \mathsf{R}'_1 = (1, 0, 0)$ and $\mathsf{G}'_1 = (0, \mathsf{m}_2, \mathsf{m}_3)$. The purpose of M_1 is to create ad-hoc inputs for E_0: given $b \in \{0, 1\}$ as input, M_1 outputs $M_1(b) = (b, \mathsf{m}_2, \mathsf{m}_3)$. [10]

• *Compose* M_1 with E_0 to obtain $E_1 = M_1 \triangleright E_0 = (1, 1, \mathsf{q}_1, \mathsf{L}_1, \mathsf{R}_1, \mathsf{G}_1)$ where: $\mathsf{q}_1 = \mathsf{q}_0 + 3$ and $\mathsf{G}_1 = (\mathsf{G}'_1 || \mathsf{G}_0)$. Let $\mathsf{L}^\star_0$ and $\mathsf{R}^\star_0$ be the string representations of the functions:

$$\mathsf{L}^\star_0(i) = \begin{cases} \mathsf{L}_0(i) + 3 & \text{if } \mathsf{L}_0(i) \neq 0, \ (i \in [\mathsf{q}_0]) \\ 0 & \text{if } \mathsf{L}_0(i) = 0, \ (i \in [\mathsf{q}_0]) \end{cases}, \quad \mathsf{R}^\star_0(i) = \begin{cases} \mathsf{R}_0(i) + 3 & \text{if } \mathsf{R}_0(i) \neq 0, \ (i \in [\mathsf{q}_0]) \\ 0 & \text{if } \mathsf{R}_0(i) = 0, \ (i \in [\mathsf{q}_0]) \end{cases}$$

The left/right input functions of E_1 are $\mathsf{L}_1 = (\mathsf{L}'_1 || \mathsf{L}^\star_0)$, $\mathsf{R}_1 = (\mathsf{R}'_1 || \mathsf{R}^\star_0)$.

Circuit composition ensures that $E_1(x_1) = E_0(x_1, \mathsf{m}_2, \mathsf{m}_3)$ and thus equation (17) implies

$$E_1(\mathsf{m}_1) = 1. \tag{18}$$

Note that E_1 can be constructed directly from E_0 given m_2 and m_3, in particular the value m_1 is not needed:

$$E_1 = \big(1, 1, \mathsf{n}_0 + 3, (100, \mathsf{L}^\star_0), (100, \mathsf{R}^\star_0), (0, \mathsf{m}_2, \mathsf{m}_3, \mathsf{G}_0)\big).$$

• *Compute* $\hat{\sigma}_1 \leftarrow \mathsf{HS.Eval}(E_1, \mathsf{pk}_1, \sigma_1)$. This is possible since E_1 is a one-input circuit. Moreover equation (18) and the correctness of the HS scheme imply that

$$\mathsf{HS.Verify}((E_1, \ell_1), \mathsf{pk}_1, \hat{\sigma}_1, 1) = 1. \tag{19}$$

Step 2. The actual inductive procedure begins now. The challenge is that $\hat{\sigma}_1$ cannot be directly checked by the verifier as it does not know the messages $\mathsf{m}_2, \mathsf{m}_3$ needed to define the circuit E_1. Our idea is to write equation (19) as $\mathsf{HS.Verify}(S_1) = 1$ for a string $S_1 = ((E_1, \ell_1), \mathsf{pk}_1, \sigma_1)$ that contains two bits that are the messages $\mathsf{m}_2, \mathsf{m}_3$, and then we want to use $\mathsf{HS.Eval}$ to create a signature proving the correctness of the computation in (19). As we shall see, this is possible by repeating our previous technique, namely seeing $\mathsf{HS.Verify}(S_1)$ as a single-input function of m_2 in which m_3 is hardwired. Repeating this approach one more time, we will later be able to let m_3 also "disappear" and use $\mathsf{HS.Eval}$ on a circuit that can be reconstructed by the verifier without knowing $\mathsf{m}_1, \mathsf{m}_2, \mathsf{m}_3$.

Coming back to this second step, in $\mathsf{MKHS.Eval}$ we proceed as follows. We define a mask circuit that outputs S_1 where the bit that embeds the value m_2 is substituted with the bit input to

[10] Recall that $\mathsf{L}'_1(1) = 1 \neq 0$ and $\mathsf{G}'_1(1) = 0$, thus the first gate outputs the input to the circuit.

mask circuit. Next, we define E_2 as the composition of this mask and the circuit HSV_2 that is the verification circuit of the signature scheme when checking the authenticity of a signature against the value 1 on input of $size(E_1)$.[11]

Facts: The position of the gate that embeds the value m_2 in E_1 is by construction $g_2 = 3\lg(N_1) + 2\mathsf{q}_1\lg(\mathsf{w}_1) + 2$, where N_1 is a given upper bound on the size of n_1 and q_1, indeed:

$$E_1 = \big(\underbrace{1, 1, \mathsf{q}_1}_{3\lg(N_1)}, \underbrace{\mathsf{L}_1, \mathsf{R}_1}_{2\mathsf{q}_1\lg(\mathsf{w}_1)}, \mathsf{G}_1 = \underbrace{(0, \mathsf{m}_2}_{2}, \mathsf{m}_3, \mathsf{G}_0)\big).$$

Since $S_1 = ((E_1, \ell_1), \mathsf{pk}_1, \sigma_1)$ is a valid input to HSV_2, we have that $|S_1| = \mathsf{n}_{\mathsf{HSV}2}$.

Given $S_1 = ((E_1, \ell_1), \mathsf{pk}_1, \hat{\sigma}_1)$ and the signature σ_2:

• Define a *mask* circuit $M_2 = (1, \mathsf{n}_{\mathsf{HSV}2}, \mathsf{n}_{\mathsf{HSV}2}, \mathsf{L}'_2, \mathsf{R}'_2, \mathsf{G}'_2)$ where

$$\mathsf{L}'_2(g) = \mathsf{R}'_2(g) = \begin{cases} 0 & \text{if } g \in [\mathsf{n}_{\mathsf{HSV}2}] \setminus \{g_2\} \\ 1 & \text{if } g = g_2 \end{cases} \text{ and } \mathsf{G}'_2(g) = \begin{cases} S_1[g] & \text{if } g \in [\mathsf{n}_{\mathsf{HSV}2}] \setminus \{g_2\} \\ 0 & \text{if } g = g_2 \end{cases}$$

The purpose of M_2 is to create ad-hoc inputs for the circuit $\mathsf{HS.Verify}$. The output of M_2 is S_1 where we overwrite the second gate of E_1 to output the value input to the circuit M_2 – instead of the constant output m_2. In particular,

$$M_2(\mathsf{m}_2) = \big(\overbrace{(1, 1, \mathsf{q}_1, \mathsf{L}_1, \mathsf{R}_1, \underbrace{(0, \mathsf{m}_2, \mathsf{m}_3, \mathsf{G}_0)}_{\mathsf{G}_1}}^{E_1}, \ell_1), \mathsf{pk}_1, \hat{\sigma}_1\big)$$

where all values should be seen as bit-strings.

• *Compose* M_2 with HSV_2 to obtain $E_2 = M_2 \triangleright \mathsf{HSV}_2 = (1, 1, \mathsf{q}_2, \mathsf{L}_2, \mathsf{R}_2, \mathsf{G}_2)$ where: $\mathsf{q}_2 = \mathsf{n}_{\mathsf{HSV}2} + \mathsf{q}_{\mathsf{HSV}_2}$; and $\mathsf{G}_2 = (\mathsf{G}'_2 || \mathsf{G}_{\mathsf{HSV}_2})$. Let $\mathsf{L}^\star_{\mathsf{HSV}_2}$ and $\mathsf{R}^\star_{\mathsf{HSV}_2}$ be the string representations of the functions:

$$\mathsf{L}^\star_{\mathsf{HSV}_2}(i) = \begin{cases} \mathsf{L}_{\mathsf{HSV}_2}(i) + \mathsf{n}_{\mathsf{HSV}_2} & \text{if } \mathsf{L}_{\mathsf{HSV}_2}(i) \neq 0,\ (i \in [\mathsf{q}_{\mathsf{HSV}_2}]) \\ 0 & \text{if } \mathsf{L}_{\mathsf{HSV}_2}(i) = 0,\ (i \in [\mathsf{q}_{\mathsf{HSV}_2}]) \end{cases},$$
$$\mathsf{R}^\star_{\mathsf{HSV}_2}(i) = \begin{cases} \mathsf{R}_{\mathsf{HSV}_2}(i) + \mathsf{n}_{\mathsf{HSV}2} & \text{if } \mathsf{R}_{\mathsf{HSV}_2}(i) \neq 0,\ (i \in [\mathsf{q}_{\mathsf{HSV}_2}]) \\ 0 & \text{if } \mathsf{R}_{\mathsf{HSV}_2}(i) = 0,\ (i \in [\mathsf{q}_{\mathsf{HSV}_2}]) \end{cases}$$

The left/right input functions of E_2 are defined as $\mathsf{L}_2 = (\mathsf{L}'_2 || \mathsf{L}^\star_{\mathsf{HSV}_2})$, $\mathsf{R}_2 = (\mathsf{R}'_2 || \mathsf{R}^\star_{\mathsf{HSV}_2})$.

Circuit composition ensures that $E_2(\mathsf{m}_2) = \mathsf{HS.Verify}((E_1, \ell_1), \mathsf{pk}_1, \hat{\sigma}_1, 1)$, therefore by (19) it holds that:

$$E_2(\mathsf{m}_2) = 1 \tag{20}$$

Note that E_2 can be constructed directly from E_0 given solely m_3, and additional public data (*i.e.*, pk_1, $\hat{\sigma}_1$, $\mathsf{HS.Verify}$, $\mathscr{P}$, y), indeed:

$$\mathsf{G}_2 = \overbrace{(1, 1, \mathsf{q}_0 + 3, \underbrace{(1, 0, 0, \mathsf{L}^\star_0)}_{\mathsf{L}_1}, \underbrace{(1, 0, 0, \mathsf{R}^\star_0)}_{\mathsf{R}_1}, \underbrace{(0, \mathbf{0}, \mathsf{m}_3, \mathsf{G}_0}_{\mathsf{G}'_1}}^{g_2-th \text{ index}}, \ell_1, \mathsf{pk}_1, \hat{\sigma}_1, \mathsf{G}_{\mathsf{HSV}_2})$$

[11] With abuse of notation $\mathsf{HSV}_2 = \mathsf{HS.Verify}((\cdot, \cdots), \cdot, \cdots, 1)$. The index $_2$ is used to keep track of the size of the input (and corresponding output) of the verification circuit.

where all values should be seen as bit-strings.

• *Compute* $\hat{\sigma}_2 \leftarrow \mathsf{HS.Eval}(E_2, \mathsf{pk}_2, \sigma_2)$. This is possible since E_2 is a one-input circuit, and σ_2 is a signature on m_2. Indeed, equation (20) and the correctness of the HS scheme imply that

$$\mathsf{HS.Verify}((E_2, \ell_2), \mathsf{pk}_2, \hat{\sigma}_2, 1) = 1. \tag{21}$$

Step 3. We proceed inductively, along the line of Step 2.

Facts: The position of the gate that embeds the value m_3 in E_2 is by construction $g_3 = 3\lg(N_2) + 2\mathsf{q}_2 \lg(\mathsf{w}_2) + g_2 + 1$, where N_2 is a given upper bound on the size of n_2 and q_2, indeed:

$$E_2 = \big(\underbrace{1, 1, \mathsf{q}_2}_{3\lg(N_2)}, \underbrace{\mathsf{L}_2, \mathsf{R}_2}_{2\mathsf{q}_2\lg(\mathsf{w}_2)}, \mathsf{G}_2 = \underbrace{(1, 1, \mathsf{q}_1, \mathsf{L}_1, \mathsf{R}_1, (0, \mathbf{0}}_{g_2}, \mathsf{m}_3, \mathsf{G}_0, \ell_1 ... \mathsf{G}_{\mathsf{HSV}_2})\big).$$

Given $S_2 = ((E_2, \ell_2), \mathsf{pk}_2, \hat{\sigma}_2)$, and the signature σ_2:

• Define a *mask* circuit $M_3 = (1, \mathsf{n}_{\mathsf{HSV}3}, \mathsf{n}_{\mathsf{HSV}3}, \mathsf{L}'_3, \mathsf{R}'_3, \mathsf{G}'_3)$ where

$$\mathsf{L}'_3(g) = \mathsf{R}'_3(g) = \begin{cases} 0 & \text{if } g \in [\mathsf{n}_{\mathsf{HSV}3}] \setminus \{g_3\} \\ 1 & \text{if } g = g_3 \end{cases} \quad \text{and } \mathsf{G}'_3(g) = \begin{cases} S_2[g] & \text{if } g \in [\mathsf{n}_{\mathsf{HSV}3}] \setminus \{g_3\} \\ 0 & \text{if } g = g_3 \end{cases}$$

The output of M_3 is S_2 where we overwrite the gate that embeds the constant value m_3 with a gate that outputs the value input to the circuit M_3. In particular, $M_3(\mathsf{m}_3) = S_2$.

• *Compose* M_3 with HSV_3 to obtain $E_3 = M_3 \triangleright \mathsf{HSV}_3 = (1, 1, \mathsf{q}_3, \mathsf{L}_3, \mathsf{R}_3, \mathsf{G}_3)$. Circuit composition ensures that $E_3(\mathsf{m}_3) = \mathsf{HS.Verify}((E_2, \ell_2), \mathsf{pk}_2, \hat{\sigma}_2, 1)$, therefore by (21) it holds that:

$$E_3(\mathsf{m}_3) = 1. \tag{22}$$

Note that E_3 can be constructed directly from $\mathcal{E}_0 = (E_0, \ell_1, \ell_2, \ell_3)$ given solely the public data $\mathsf{pk}_i, \hat{\sigma}_i$ for $i \in [2]$: indeed:

$$E_3 = \Big(1, 1, \mathsf{n}_{\mathsf{HSV}3} + \mathsf{q}_{\mathsf{HSV}_3}, \mathsf{L}_3 = (\underbrace{0..0..1}_{g_3}0..0, \overbrace{\mathsf{L}^\star_{\mathsf{HSV}_3}}^{\mathsf{q}_{\mathsf{HSV}_3}}), \mathsf{R}_3 = (\underbrace{0..0..1}_{g_3}0..0, \overbrace{\mathsf{R}^\star_{\mathsf{HSV}_3}}^{\mathsf{q}_{\mathsf{HSV}_3}}),$$

$$\underbrace{(\overbrace{0..01}^{g_2}0..0, \mathsf{L}^\star_{\mathsf{HSV}_2})}$$

$$\mathsf{G}_3 = \big((1, 1, \mathsf{n}_{\mathsf{HSV}2} + \mathsf{q}_{\mathsf{HSV}_2},\ \mathsf{L}_2,\ \mathsf{R}_2, \mathsf{string}, \mathsf{G}_{\mathsf{HSV}_2}, \ell_2), \mathsf{pk}_2, \hat{\sigma}_2, \mathsf{G}_{\mathsf{HSV}_3})\big)\Big)$$

$$\overbrace{(1, 1, \mathsf{q}_0 + 3, \underbrace{(1, 0, 0, \mathsf{L}^\star_0)}_{\mathsf{L}_1}, \underbrace{(1, 0, 0, \mathsf{R}^\star_0)}_{\mathsf{R}_1}, (0, 0, 0, \mathsf{G}_0, \ell_1, \mathsf{pk}_1, \hat{\sigma}_1))}$$

where all values should be seen as bit-strings.

• *Compute* $\hat{\sigma}_3 \leftarrow \mathsf{HS.Eval}(E_3, \mathsf{pk}_3, \sigma_3)$. This is possible since E_3 is a one-input circuit. From Equation (22) and the correctness of the HS scheme we get:

$$\mathsf{HS.Verify}((E_3, \ell_3), \mathsf{pk}_3, \hat{\sigma}_3, 1) = 1. \tag{23}$$

The multi-key homomorphic evaluation algorithm outputs $\hat{\sigma} = (\hat{\sigma}_1, \hat{\sigma}_2, \hat{\sigma}_3)$.

In order to verify $\hat{\sigma}$ the verifier simply needs the labeled program $\mathscr{P} = (C, (\ell_1, \ell_2, \ell_3))$, the value y corresponding to the claimed output of $\mathscr{P}$, the three public keys pk_i for $i \in [3]$ and the multi-key homomorphic signature $\hat{\sigma} = (\hat{\sigma}_1, \hat{\sigma}_2, \hat{\sigma}_3)$. The verification process begins by constructing the circuit E_0. As noted before, this can be done given solely C and y. Next, the verifier computes directly the circuit E_3. This can be done using the available values $\ell_i, \mathsf{pk}_i, \hat{\sigma}_i$ for $i \in [2]$ and the (public) circuit descriptions of HSV_2 and HSV_3. Let $\mathscr{E}_3 = (E_3, \ell_3)$, the verification concludes by running:

$$\mathsf{HS.Verify}(\mathscr{E}_3, \mathsf{pk}_3, \hat{\sigma}_3, 1)$$

It is easy to see that by correctness, this returns 1, as stated in Equation (23).