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动力学与控制

Caputo 导数下分数阶 Birkhoff 系统的准对称性与分数阶 Noether 定理[1)]

张　毅[2)]

(苏州科技大学土木工程学院，江苏苏州 215011)

摘要　应用分数阶模型可以更准确地描述和研究复杂系统的动力学行为和物理过程，同时 Birkhoff 力学是 Hamilton 力学的推广，因此研究分数阶 Birkhoff 系统动力学具有重要意义．分数阶 Noether 定理揭示了 Noether 对称变换与分数阶守恒量之间的内在联系，但是当变换拓展为 Noether 准对称变换时，该定理的推广遇到了很大的困难．本文基于时间重新参数化方法提出并研究 Caputo 导数下分数阶 Birkhoff 系统的 Noether 准对称性与守恒量．首先，将时间重新参数化方法应用于经典 Birkhoff 系统的 Noether 准对称性与守恒量研究，建立了相应的 Noether 定理；其次，基于分数阶 Pfaff 作用量分别在时间不变的和一般单参数无限小变换群下的不变性给出分数阶 Birkhoff 系统的 Noether 准对称变换的定义和判据，基于 Frederico 和 Torres 提出的分数阶守恒量定义，利用时间重新参数化方法建立了分数阶 Birkhoff 系统的 Noether 定理，从而揭示了分数阶 Birkhoff 系统的 Noether 准对称性与分数阶守恒量之间的内在联系．分数阶 Birkhoff 系统的 Noether 对称性定理和经典 Birkhoff 系统的 Noether 定理是其特例．最后以分数阶 Hojman-Urrutia 问题为例说明结果的应用．

关键词　分数阶 Birkhoff 系统，Noether 准对称性，Frederico-Torres 分数阶守恒量，Caputo 导数

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QUASI-SYMMETRY AND NOETHER'S THEOREM FOR FRACTIONAL BIRKHOFFIAN SYSTEMS IN TERMS OF CAPUTO DERIVATIVES[1)]

Zhang Yi[2)]

(*College of Civil Engineering*，*Suzhou University of Science and Technology*，*Suzhou* 215011，*Jiangshu*，*China*)

Abstract　The dynamical behavior and physical process of a complex system can be described and studied more accurately by using a fractional model, at the same time the Birkhoffian mechanics is a generalization of Hamiltonian mechanics, and therefore, the study of dynamics of fractional Birkhoffian systems is of great significance. Fractional Noether's theorem reveals the intrinsic relation between the Noether symmetric transformation and the fractional conserved quantity, but when the transformation is replaced by the Noether quasi-symmetric transformation, the corresponding extension of Noether's theorem is very difficult. In this paper, the Noether quasi-symmetry and the conserved quantity for fractional Birkhoffian systems in terms of Caputo derivatives are presented and studied by using a technique of time-reparametrization. Firstly, the technique is applied to the study of the Noether quasi-symmetry and the conserved quantity

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2) 张毅, 教授, 主要研究方向: 分析力学. E-mail: zhy@mail.usts.edu.cn

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for classical Birkhoffian systems and Noether's theorem in its general form is established. Secondly, the definitions and criteria of Noether quasi-symmetric transformations for fractional Birkhoffian systems are given which are based on the invariance of fractional Pfaff action under one-parameter infinitesimal group of transformations without transforming the time and with transforming the time, respectively. Based on the definition of fractional conserved quantity proposed by Frederico and Torres, Noether's theorem for fractional Birkhoffian systems is established by using the method of time-reparametrization. The theorem reveals the inner relationship between Noether quasi-symmetry and fractional conserved quantity and contains Noether's theorem for the symmetry of fractional Birkhoffian system and Noether's theorem for classical Birkhoffian system as its specials. Finally, we take the Hojman-Urrutia problem as an example to illustrate the application of the results.

Key words fractional Birkhoffian system, Noether quasi-symmetry, Frederico-Torres fractional conserved quantity, caputo derivative

引 言

分数阶微积分由于具有记忆性和非局域性等特点，近四十年来被广泛地应用于解决科学和工程的许多领域的各种问题 [1-3]. 1996 年，Riewe[4-5] 利用分数阶微积分将非保守力纳入 Lagrange 函数和 Hamilton 函数中构建了非保守动力学系统的分数阶模型，首次提出并初步研究了分数阶变分问题. 在此基础上，Agrawal[6-7]，Baleanu 等 [8-9]，Atanacković 等 [10-11]，Almeida 等 [12-13] 对分数阶变分问题进行了深入研究. Frederico 等 [14-18] 最早开展分数阶 Noether 对称性与守恒量的研究，提出了分数阶守恒量的定义，利用时间重新参数化方法，建立了分数阶 Noether 定理. 近年来，分数阶 Noether 对称性与守恒量的研究取得了重要进展 [19-31]. 基于 Frederico-Torres 分数阶守恒量定义建立的 Noether 定理揭示了 Noether 对称变换与分数阶守恒量之间的内在联系，但是当其变换拓展为 Noether 准对称变换时，该定理的推广遇到了很大的困难. 迄今为止笔者尚未见到关于 Noether 准对称变换与 Frederico-Torres 分数阶守恒量的 Noether 定理的研究报道. 本文提出并研究 Caputo 导数下分数阶 Birkhoff 系统的 Noether 准对称性与守恒量问题，基于 Frederico-Torres 分数阶守恒量定义，给出 Noether 准对称性的定义和判据，利用时间重新参数化方法，研究 Noether 准对称性与 Frederico-Torres 分数阶守恒量的联系，建立了更一般的分数阶 Noether 定理.

1 分数阶导数及其性质

本节列出文中涉及的分数阶导数的定义及相关性质，详细的证明和讨论可参见文献 [2-3].

设函数 $f(t)$ 和 $g(t)$ 在区间 $[a,b]$ 上连续可积，则 Riemann-Liouville 分数阶左导数定义为

$$ {}_aD_t^\alpha f(t) = \frac{1}{\Gamma(1-\alpha)} \frac{\mathrm{d}}{\mathrm{d}t} \int_a^t (t-\tau)^{-\alpha} f(\tau)\,\mathrm{d}\tau \tag{1}$$

Riemann–Liouville 分数阶右导数为

$$ {}_tD_b^\alpha f(t) = -\frac{1}{\Gamma(1-\alpha)} \frac{\mathrm{d}}{\mathrm{d}t} \int_t^b (\tau-t)^{-\alpha} f(\tau)\,\mathrm{d}\tau \tag{2}$$

Caputo 分数阶左导数定义为

$$ {}_a^{\mathrm{C}}D_t^\alpha f(t) = \frac{1}{\Gamma(1-\alpha)} \int_a^t (t-\tau)^{-\alpha} \frac{\mathrm{d}}{\mathrm{d}\tau} f(\tau)\,\mathrm{d}\tau \tag{3}$$

Caputo 分数阶右导数定义为

$$ {}_t^{\mathrm{C}}D_b^\alpha f(t) = -\frac{1}{\Gamma(1-\alpha)} \int_t^b (\tau-t)^{-\alpha} \frac{\mathrm{d}}{\mathrm{d}\tau} f(\tau)\,\mathrm{d}\tau \tag{4}$$

其中，$\Gamma(*)$ 是 Euler-Gamma 函数，α 是导数的阶，且 $0 \leqslant \alpha < 1$.

设函数 $f(t)$ 和 $g(t)$ 为在区间 $[a,b]$ 上的光滑函数，且 $f(a) = f(b) = 0$，则 Caputo 导数下的分数阶分部积分公式为

$$ \int_a^b g(t)\,{}_a^{\mathrm{C}}D_t^\alpha f(t)\mathrm{d}t = \int_a^b f(t)\,{}_tD_b^\alpha g(t)\mathrm{d}t \tag{5}$$

和

$$ \int_a^b g(t)\,{}_t^{\mathrm{C}}D_b^\alpha f(t)\mathrm{d}t = \int_a^b f(t)\,{}_aD_t^\alpha g(t)\mathrm{d}t \tag{6}$$

设 $0 < \alpha < 1$ 和 $0 < \beta < 1$，且 $f'(a) = 0$，则有

$$ {}_a^{\mathrm{C}}D_t^{\alpha}\,{}_a^{\mathrm{C}}D_t^{\beta} f(t) = {}_a^{\mathrm{C}}D_t^{\beta}\,{}_a^{\mathrm{C}}D_t^{\alpha} f(t) = {}_a^{\mathrm{C}}D_t^{\alpha+\beta} f(t) \tag{7}$$

在 Caputo 导数下，定义算子 ${}^{\mathrm{C}}\mathcal{D}_t^\gamma(f,g)$ 为

$$ {}^{\mathrm{C}}\mathcal{D}_t^\gamma(f,g) = -g\,{}_tD_b^\gamma f + f\,{}_a^{\mathrm{C}}D_t^\gamma g \tag{8}$$

或

$$ {}^{\mathrm{C}}\mathcal{D}_t^{\gamma}(f,g) = -g_t^{\mathrm{C}}D_b^{\gamma}f + f_a D_t^{\gamma}g \tag{9}$$

当 $\gamma = 1$ 时，式 (8) 和式 (9) 成为

$$ {}^{\mathrm{C}}\mathcal{D}_t^{1}(f,g) = -g_t D_b^{1}f + f_a^{\mathrm{C}}D_t^{1}g = g\dot{f} + f\dot{g} = \frac{\mathrm{d}}{\mathrm{d}t}(fg) \tag{10}$$

$$ {}^{\mathrm{C}}\mathcal{D}_t^{1}(f,g) = -g_t^{\mathrm{C}}D_b^{1}f + f_a D_t^{1}g = g\dot{f} + f\dot{g} = \frac{\mathrm{d}}{\mathrm{d}t}(fg) \tag{11}$$

此时 ${}^{\mathrm{C}}\mathcal{D}_t^{1}(f,g) = {}^{\mathrm{C}}\mathcal{D}_t^{1}(g,f)$，但是一般情况下 ${}^{\mathrm{C}}\mathcal{D}_t^{\gamma}(f,g) \neq {}^{\mathrm{C}}\mathcal{D}_t^{\gamma}(g,f)$.

2 经典 Birkhoff 系统的 Noether 准对称性与守恒量

积分泛函

$$ A[a^{\nu}(\cdot)] = \int_{t_1}^{t_2}\left[R_{\mu}(t,a^{\nu})\dot{a}^{\mu} - B(t,a^{\nu})\right]\mathrm{d}t \tag{12}$$

称为 Pfaff 作用量，其中，$B(t,a^{\nu})$ 是 Birkhoff 函数，$R_{\mu}(t,a^{\nu})\ (\mu = 1,2,\cdots,2n)$ 是 Birkhoff 函数组.

Pfaff-Birkhoff 原理可表示为 [32]

$$ \delta A = 0 \tag{13}$$

带有交换关系

$$ \mathrm{d}\delta a^{\mu} = \delta\mathrm{d}a^{\mu} \quad (\mu = 1,2,\cdots,2n) \tag{14}$$

及端点条件

$$ \delta a^{\mu}|_{t=t_1} = \delta a^{\mu}|_{t=t_2} = 0 \quad (\mu = 1,2,\cdots,2n) \tag{15}$$

由式 (13) ~ 式 (15) 容易导出 Birkhoff 方程

$$ \left(\frac{\partial R_{\nu}}{\partial a^{\mu}} - \frac{\partial R_{\mu}}{\partial a^{\nu}}\right)\dot{a}^{\nu} - \left(\frac{\partial B}{\partial a^{\mu}} + \frac{\partial R_{\mu}}{\partial t}\right) = 0 \quad (\mu = 1,2,\cdots,2n) \tag{16}$$

其中, 矩阵

$$ (\omega_{\mu\nu}) = \left(\frac{\partial R_{\nu}}{\partial a^{\mu}} - \frac{\partial R_{\mu}}{\partial a^{\nu}}\right)$$

是非退化的. 由 Pfaff-Birkhoff 原理 (式 (13) ~ 式 (15)) 和 Birkhoff 方程 (16) 确定的动力学系统称为经典 Birkhoff 系统.

下面利用时间重新参数化方法研究经典 Birkhoff 系统的 Noether 准对称性与守恒量. 首先，研究时间不变的无限小变换下经典 Birkhoff 系统的 Noether 准对称性.

定义 1　对于经典 Birkhoff 系统，设 R_{μ}^{1} 和 B^{1} 是另外的 Birkhoff 函数组和 Birkhoff 函数，如果在时间不变的单参数无限小变换群

$$ \bar{a}^{\mu}(t) = a^{\mu}(t) + \varepsilon\xi_{\mu}(t,a^{\nu}) + o(\varepsilon) \quad (\mu = 1,2,\cdots,2n) \tag{17}$$

作用下，对任意 $[T_1,T_2] \subseteq [t_1,t_2]$，满足如下条件

$$ \int_{T_1}^{T_2}\left[R_{\mu}(t,a^{\nu})\dot{a}^{\mu} - B(t,a^{\nu})\right]\mathrm{d}t = \int_{T_1}^{T_2}\left[R_{\mu}^{1}(t,\bar{a}^{\nu})\dot{\bar{a}}^{\mu} - B^{1}(t,\bar{a}^{\nu})\right]\mathrm{d}t \tag{18}$$

其中，ε 为无限小参数，ξ_{μ} 是无限小变换的生成元，则称 Pfaff 作用量 (式 (12)) 在无限小变换 (式 (17)) 下准不变的，变换 (式 (17)) 为系统在时间不变的无限小变换下的 Noether 准对称变换.

由式 (18) 确定的 R_{μ}^{1}，B^{1} 和 R_{μ}，B 具有相同的运动微分方程，此时有 [32]

$$ \int_{T_1}^{T_2}\left[R_{\mu}(t,a^{\nu})\dot{a}^{\mu} - B(t,a^{\nu})\right]\mathrm{d}t = \int_{T_1}^{T_2}\left[R_{\mu}(t,\bar{a}^{\nu})\dot{\bar{a}}^{\mu} - B(t,\bar{a}^{\nu}) + \frac{\mathrm{d}}{\mathrm{d}t}\Delta G(t,\bar{a}^{\nu})\right]\mathrm{d}t \tag{19}$$

其中，$\Delta G = \varepsilon G(t,a^{\nu})$，函数 $G(t,a^{\nu})$ 称为规范函数.

判据 1　对于经典 Birkhoff 系统，如果存在规范函数 $G(t,a^{\nu})$ 使得无限小变换 (式 (17)) 的生成元 ξ_{μ} 满足条件

$$ \frac{\partial R_{\mu}}{\partial a^{\nu}}\xi_{\nu}\dot{a}^{\mu} + R_{\mu}\dot{\xi}_{\mu} - \frac{\partial B}{\partial a^{\mu}}\xi_{\mu} + \frac{\mathrm{d}}{\mathrm{d}t}G = 0 \tag{20}$$

则变换 (式 (17)) 是系统在时间不变的无限小变换下的 Noether 准对称变换.

证明　由于积分区间 $[T_1,T_2]$ 的任意性，式 (19) 等价于

$$ R_{\mu}(t,a^{\nu})\dot{a}^{\mu} - B(t,a^{\nu}) = R_{\mu}(t,\bar{a}^{\nu})\dot{\bar{a}}^{\mu} - B(t,\bar{a}^{\nu}) + \frac{\mathrm{d}}{\mathrm{d}t}\Delta G(t,\bar{a}^{\nu}) \tag{21}$$

将方程 (21) 等号两边对 ε 求导，并令 $\varepsilon = 0$，立即得到式 (20). 证毕.

定理 1　对于经典 Birkhoff 系统，如果时间不变的无限小变换 (式 (17)) 是系统的 Noether 准对称变换，则

$$ I_N = R_{\mu}\xi_{\mu} + G \tag{22}$$

是系统的一个守恒量.

证明 利用条件 (式 (20)) 和方程 (16)，有

$$\begin{aligned}\frac{\mathrm{d}I_N}{\mathrm{d}t} &= \frac{\mathrm{d}R_\mu}{\mathrm{d}t}\xi_\mu + R_\mu\frac{\mathrm{d}\xi_\mu}{\mathrm{d}t} + \frac{\mathrm{d}G}{\mathrm{d}t} = \\ &\left(\frac{\partial R_\mu}{\partial t} + \frac{\partial R_\mu}{\partial a^\nu}\dot{a}^\nu\right)\xi_\mu - \frac{\partial R_\mu}{\partial a^\nu}\xi_\nu\dot{a}^\mu + \frac{\partial B}{\partial a^\mu}\xi_\mu = \\ &-\left(\frac{\partial R_\nu}{\partial a^\mu} - \frac{\partial R_\mu}{\partial a^\nu}\right)\dot{a}^\nu\xi_\mu + \frac{\partial B}{\partial a^\mu}\xi_\mu + \frac{\partial R_\mu}{\partial t}\xi_\mu = 0\end{aligned}$$

因此，系统存在守恒量 (式 (22)). 证毕.

其次，研究一般无限小变换下经典 Birkhoff 系统的 Noether 准对称性.

定义 2 对于经典 Birkhoff 系统，设 R_μ^1 和 B^1 是另外的 Birkhoff 函数组和 Birkhoff 函数，如果在一般单参数无限小变换群

$$\left.\begin{aligned}&\bar{t} = t + \varepsilon\xi_0(t, a^\nu) + o(\varepsilon)\\ &\bar{a}^\mu(t) = a^\mu(t) + \varepsilon\xi_\mu(t, a^\nu) + o(\varepsilon)\\ &(\mu = 1, 2, \cdots, 2n)\end{aligned}\right\} \tag{23}$$

作用下，对任意 $[T_1, T_2] \subseteq [t_1, t_2]$，满足如下条件

$$\begin{aligned}&\int_{T_1}^{T_2}\left[R_\mu(t, a^\nu)\dot{a}^\mu - B(t, a^\nu)\right]\mathrm{d}t = \\ &\qquad\int_{\bar{T}_1}^{\bar{T}_2}\left[R_\mu^1(\bar{t}, \bar{a}^\nu)\dot{\bar{a}}^\mu - B^1(\bar{t}, \bar{a}^\nu)\right]\mathrm{d}\bar{t}\end{aligned} \tag{24}$$

其中，ε 为无限小参数，ξ_0，ξ_μ 是无限小变换的生成元，则称 Pfaff 作用量 (式 (12)) 在无限小变换 (式 (23)) 下准不变的，变换 (式 (23)) 为系统在一般无限小变换下的 Noether 准对称变换.

由式 (24) 确定的 R_μ^1，B^1 和 R_μ，B 具有相同的运动微分方程，此时有

$$\begin{aligned}&\int_{T_1}^{T_2}\left[R_\mu(t, a^\nu)\dot{a}^\mu - B(t, a^\nu)\right]\mathrm{d}t = \\ &\qquad\int_{\bar{T}_1}^{\bar{T}_2}\left[R_\mu(\bar{t}, \bar{a}^\nu)\dot{\bar{a}}^\mu - B(\bar{t}, \bar{a}^\nu) + \frac{\mathrm{d}}{\mathrm{d}\bar{t}}\Delta G(\bar{t}, \bar{a}^\nu)\right]\mathrm{d}\bar{t}\end{aligned} \tag{25}$$

于是有

判据 2 对于经典 Birkhoff 系统，如果存在规范函数 $G(t, a^\nu)$ 使得无限小变换 (式 (23)) 的生成元 ξ_0，ξ_μ 满足条件

$$\begin{aligned}&\left(\frac{\partial R_\mu}{\partial t}\xi_0 + \frac{\partial R_\mu}{\partial a^\nu}\xi_\nu\right)\dot{a}^\mu + R_\mu\dot{\xi}_\mu - \\ &\qquad\frac{\partial B}{\partial t}\xi_0 - \frac{\partial B}{\partial a^\nu}\xi_\nu - B\dot{\xi}_0 + \frac{\mathrm{d}}{\mathrm{d}t}G = 0\end{aligned} \tag{26}$$

则变换 (式 (23)) 是系统在一般无限小变换下的 Noether 准对称变换.

证明 由式 (25) 可得出

$$\begin{aligned}&\int_{T_1}^{T_2}\left[R_\mu(t, a^\nu)\dot{a}^\mu - B(t, a^\nu)\right]\mathrm{d}t = \\ &\quad\int_{T_1}^{T_2}\left[R_\mu(\bar{t}, \bar{a}^\nu)\frac{\mathrm{d}\bar{a}^\mu}{\mathrm{d}t} - B(\bar{t}, \bar{a}^\nu)\frac{\mathrm{d}\bar{t}}{\mathrm{d}t} + \frac{\mathrm{d}}{\mathrm{d}t}\Delta G(\bar{t}, \bar{a}^\nu)\right]\mathrm{d}t\end{aligned} \tag{27}$$

由积分区间 $[T_1, T_2]$ 的任意性，式 (27) 等价于

$$\begin{aligned}&R_\mu(t, a^\nu)\dot{a}^\mu - B(t, a^\nu) = \\ &\qquad R_\mu(\bar{t}, \bar{a}^\nu)\frac{\mathrm{d}\bar{a}^\mu}{\mathrm{d}t} - B(\bar{t}, \bar{a}^\nu)\frac{\mathrm{d}\bar{t}}{\mathrm{d}t} + \frac{\mathrm{d}}{\mathrm{d}t}\Delta G(\bar{t}, \bar{a}^\nu)\end{aligned} \tag{28}$$

将方程 (28) 等号两边对 ε 求导，并令 $\varepsilon = 0$，立即得到式 (26). 证毕.

定理 2 对于经典 Birkhoff 系统，如果一般无限小变换 (式 (23)) 是系统的 Noether 准对称变换，则

$$I_N = R_\mu\xi_\mu - B\xi_0 + G \tag{29}$$

是系统的一个守恒量.

证明 如果将 t 看作为一个独立变量，则每个非自治问题 (式 (12)) 等价于一个自治问题 [14]. 事实上，设

$$L(t, a^\nu(t), \dot{a}^\nu(t)) = R_\mu(t, a^\nu(t))\dot{a}^\mu(t) - B(t, a^\nu(t)) \tag{30}$$

则在时间 t 的一一对应的李普希茨变换

$$[t_1, t_2] \ni t \mapsto \sigma \in [\sigma_1, \sigma_2] \tag{31}$$

作用下，有

$$\begin{aligned}&A[a^\nu(\cdot)] = \int_{t_1}^{t_2} L(t, a^\nu(t), \dot{a}^\nu(t))\mathrm{d}t = \\ &\quad\int_{\sigma_1}^{\sigma_2}\left[R_\mu(t(\sigma), a^\nu(t(\sigma)))\frac{\mathrm{d}a^\mu(t(\sigma))}{\mathrm{d}t(\sigma)} - \right.\\ &\quad\left.B(t(\sigma), a^\nu(t(\sigma)))\right]\frac{\mathrm{d}t(\sigma)}{\mathrm{d}\sigma}\mathrm{d}\sigma = \\ &\quad\int_{\sigma_1}^{\sigma_2}\left[R_\mu(t(\sigma), a^\nu(t(\sigma)))\frac{\mathrm{d}a^\mu(t(\sigma))}{\mathrm{d}\sigma} - \right.\\ &\quad\left.B(t(\sigma), a^\nu(t(\sigma)))\frac{\mathrm{d}t(\sigma)}{\mathrm{d}\sigma}\right]\mathrm{d}\sigma = \\ &\quad\int_{\sigma_1}^{\sigma_2}\bar{L}(t(\sigma), a^\nu(t(\sigma)), t'(\sigma), (a^\nu)'(t(\sigma)))\mathrm{d}\sigma = \\ &\quad\bar{A}[t(\cdot), a^\nu(t(\cdot))]\end{aligned} \tag{32}$$

其中

$$t(\sigma_1)=t_1,\ t(\sigma_2)=t_2$$

$$t'(\sigma)=\frac{\mathrm{d}t(\sigma)}{\mathrm{d}\sigma},\ (a^\nu)'(t(\sigma))=\frac{\mathrm{d}a^\nu(t(\sigma))}{\mathrm{d}\sigma}$$

因此，如果作用量 $A[a^\nu(\cdot)]$ 在定义 2 意义下是准不变的，则作用量 $\bar{A}[t(\cdot),a^\nu(t(\cdot))]$ 在定义 1 意义下是准不变的．由定理 1 可得到

$$I_N=R_\mu\xi_\mu+(-B)\xi_0+G$$

是系统的一个守恒量．证毕．

定理 1 和定理 2 可称为经典 Birkhoff 系统的 Noether 定理．该定理揭示了经典 Birkhoff 系统分别在时间不变和一般单参数无限小变换下的 Noether 准对称性与守恒量之间的内在联系．如果 $\dot{G}\equiv 0$，则定理给出了系统的 Noether 对称性与守恒量的联系．不同于以往的工作 [32-34]，这里的证明采用了时间重新参数化方法：首先，在时间不变的无限小变换下给出守恒量 (式 (22))；其次，引入李普希茨变换，将一个非自治问题化为一个自治问题，得到一般无限小变换下的守恒量 (式 (29))．

3 分数阶 Birkhoff 系统的 Noether 准对称性与守恒量

设基于 Caputo 导数的分数阶 Pfaff 作用量为

$$S[a^\mu(\cdot)]=\int_{t_1}^{t_2}\left[R_\mu(t,a^\nu)\,{}_{t_1}^{\mathrm{C}}D_t^\alpha a^\mu-B(t,a^\nu)\right]\mathrm{d}t \tag{33}$$

则分数阶 Pfaff-Birkhoff 原理可表示为

$$\delta S=0 \tag{34}$$

带有交换关系

$$\left.\begin{aligned}{}_{t_1}^{\mathrm{C}}D_t^\alpha\delta a^\mu&=\delta\,{}_{t_1}^{\mathrm{C}}D_t^\alpha a^\mu\\ {}_{t}^{\mathrm{C}}D_{t_2}^\alpha\delta a^\mu&=\delta\,{}_{t}^{\mathrm{C}}D_{t_2}^\alpha a^\mu\end{aligned}\quad(\mu=1,2,\cdots,2n)\right\} \tag{35}$$

以及端点条件

$$\delta a^\mu|_{t=t_1}=\delta a^\mu|_{t=t_2}=0\quad(\mu=1,2,\cdots,2n) \tag{36}$$

由式 (34) ~ 式 (36) 可导出 Caputo 导数下分数阶 Birkhoff 方程

$$\frac{\partial R_\nu}{\partial a^\mu}\,{}_{t_1}^{\mathrm{C}}D_t^\alpha a^\nu-\frac{\partial B}{\partial a^\mu}+{}_tD_{t_2}^\alpha R_\mu=0 \tag{37}$$

由分数阶 Pfaff-Birkhoff 原理 (式 (34) ~ 式 (36)) 和分数阶 Birkhoff 方程 (37) 确定的动力学系统称为 Caputo 导数下分数阶 Birkhoff 系统．

下面研究 Caputo 导数下分数阶 Birkhoff 系统的 Noether 准对称性与守恒量问题．首先基于由 Frederico 和 Torres 给出的分数阶守恒量概念 [14]，建立分数阶 Birkhoff 系统的分数阶守恒量定义，有

定义 3　对于 Caputo 导数下分数阶 Birkhoff 系统，当且仅当沿着分数阶 Birkhoff 方程 (37) 的所有解曲线，有

$$I\left(t,a^\nu,{}_{t_1}^{\mathrm{C}}D_t^\alpha a^\nu\right)=\sum_{i=1}^{m}I_i^1\left(t,a^\nu,{}_{t_1}^{\mathrm{C}}D_t^\alpha a^\nu\right)\cdot I_i^2\left(t,a^\nu,{}_{t_1}^{\mathrm{C}}D_t^\alpha a^\nu\right) \tag{38}$$

其中 m 是任意整数，对于每一组函数 I_i^1 和 I_i^2 $(i=1,2,\cdots,m)$，满足

$${}^{\mathrm{C}}\mathcal{D}_t^\alpha\left(I_i^{j_i^1}\left(t,a^\nu,{}_{t_1}^{\mathrm{C}}D_t^\alpha a^\nu\right),I_i^{j_i^2}\left(t,a^\nu,{}_{t_1}^{\mathrm{C}}D_t^\alpha a^\nu\right)\right)=0 \tag{39}$$

这里 $j_i^1=1$ 和 $j_i^2=2$ 或 $j_i^1=2$ 和 $j_i^2=1$，则 $I\left(t,a^\nu,{}_{t_1}^{\mathrm{C}}D_t^\alpha a^\nu\right)$ 是该系统的分数阶守恒量．

定义 3 称为 Caputo 导数下分数阶 Birkhoff 系统的 Frederico-Torres 分数阶守恒量定义．

其次，研究时间不变的无限小变换下分数阶 Birkhoff 系统的 Noether 准对称性与守恒量．

定义 4　对于 Caputo 导数下分数阶 Birkhoff 系统，设 R_μ^1 和 B^1 是另外的 Birkhoff 函数组和 Birkhoff 函数，如果在时间不变的单参数无限小变换群 (式 (17)) 作用下，对任意 $[T_1,T_2]\subseteq[t_1,t_2]$，满足如下条件

$$\int_{T_1}^{T_2}\left[R_\mu(t,a^\nu)\,{}_{t_1}^{\mathrm{C}}D_t^\alpha a^\mu-B(t,a^\nu)\right]\mathrm{d}t=\int_{T_1}^{T_2}\left[R_\mu^1(t,\bar{a}^\nu)\,{}_{t_1}^{\mathrm{C}}D_t^\alpha\bar{a}^\mu-B^1(t,\bar{a}^\nu)\right]\mathrm{d}t \tag{40}$$

则称分数阶 Pfaff 作用量 (式 (33)) 在无限小变换 (式 (17)) 下是准不变的，变换 (式 (17)) 为系统在时间不变的无限小变换下的 Noether 准对称变换．

由式 (40) 确定的 R_μ^1，B^1 和 R_μ，B 具有相同的运动微分方程，此时有

$$\int_{T_1}^{T_2}\left[R_\mu(t,a^\nu)\,{}_{t_1}^{\mathrm{C}}D_t^\alpha a^\mu-B(t,a^\nu)\right]\mathrm{d}t=\int_{T_1}^{T_2}\left[R_\mu(t,\bar{a}^\nu)\,{}_{t_1}^{\mathrm{C}}D_t^\alpha\bar{a}^\mu-B(t,\bar{a}^\nu)+\frac{\mathrm{d}}{\mathrm{d}t}\Delta G(t,\bar{a}^\nu)\right]\mathrm{d}t \tag{41}$$

其中 $\Delta G=\varepsilon G(t,a^\nu)$．

判据 3 对于 Caputo 导数下分数阶 Birkhoff 系统，如果存在规范函数 $G(t,a^\nu)$ 使得无限小变换 (式 (17)) 的生成元 ξ_μ 满足条件

$$\frac{\partial R_\mu}{\partial a^\nu}\xi_\nu {}_{t_1}^{C}D_t^\alpha a^\mu + R_\mu {}_{t_1}^{C}D_t^\alpha \xi_\mu - \frac{\partial B}{\partial a^\nu}\xi_\nu + \frac{d}{dt}G = 0 \tag{42}$$

则变换 (式 (17)) 是系统在时间不变的无限小变换下的 Noether 准对称变换.

证明 由于积分区间 $[T_1,T_2]$ 的任意性，式 (41) 等价于

$$R_\mu(t,a^\nu){}_{t_1}^{C}D_t^\alpha a^\mu - B(t,a^\nu) = R_\mu(t,\bar{a}^\nu){}_{t_1}^{C}D_t^\alpha \bar{a}^\mu - B(t,\bar{a}^\nu) + \frac{d}{dt}\Delta G(t,\bar{a}^\nu) \tag{43}$$

将方程 (43) 等号两边对 ε 求导，并令 $\varepsilon=0$，我们有

$$0 = \frac{\partial R_\mu}{\partial a^\nu}\xi_\nu {}_{t_1}^{C}D_t^\alpha a^\mu + R_\mu \frac{d}{d\varepsilon}\left\{\frac{1}{\Gamma(1-\alpha)}\cdot \int_{t_1}^{t}(t-\tau)^{-\alpha}\frac{d}{d\tau}\left[a^\mu(\tau)+\varepsilon\xi_\mu(\tau,a^\nu)\right]d\tau\right\}\Bigg|_{\varepsilon=0} - \frac{\partial B}{\partial a^\nu}\xi_\nu + \frac{d}{dt}G = \frac{\partial R_\mu}{\partial a^\nu}\xi_\nu {}_{t_1}^{C}D_t^\alpha a^\mu + R_\mu {}_{t_1}^{C}D_t^\alpha \xi_\mu - \frac{\partial B}{\partial a^\nu}\xi_\nu + \frac{d}{dt}G$$

证毕.

定理 3 对于 Caputo 导数下分数阶 Birkhoff 系统，如果时间不变的无限小变换 (式 (17)) 是系统的 Noether 准对称变换，则

$$I_N = R_\mu\xi_\mu + {}_{t_1}D_t^{1-\alpha}G \tag{44}$$

是系统在定义 3 意义下的一个分数阶守恒量.

证明 由方程 (37)，得

$$\frac{\partial B}{\partial a^\mu} = \frac{\partial R_\nu}{\partial a^\mu}{}_{t_1}^{C}D_t^\alpha a^\nu + {}_tD_{t_2}^\alpha R_\mu \tag{45}$$

将式 (45) 代入式 (42)，并注意到式 (8) 和式 (9)，我们有

$$0 = \frac{\partial R_\mu}{\partial a^\nu}\xi_\nu {}_{t_1}^{C}D_t^\alpha a^\mu + R_\mu {}_{t_1}^{C}D_t^\alpha \xi_\mu - \frac{\partial R_\nu}{\partial a^\mu}\xi_\mu {}_{t_1}^{C}D_t^\alpha a^\nu - \xi_\mu {}_tD_{t_2}^\alpha R_\mu + \frac{d}{dt}G = -\xi_\mu {}_tD_{t_2}^\alpha R_\mu + R_\mu {}_{t_1}^{C}D_t^\alpha \xi_\mu + \frac{d}{dt}G = {}^{C}\mathcal{D}_t^\alpha\left(R_\mu,\xi_\mu\right) + {}^{C}\mathcal{D}_t^\alpha\left(1,{}_{t_1}D_t^{1-\alpha}G\right)$$

由定义 3，得守恒量式 (44). 证毕.

最后，研究一般无限小变换下分数阶 Birkhoff 系统的 Noether 准对称性与守恒量.

定义 5 对于 Caputo 导数下分数阶 Birkhoff 系统，设 R_μ^1 和 B^1 是另外的 Birkhoff 函数组和 Birkhoff 函数，如果在一般单参数无限小变换群 (式 (23)) 作用下，对任意 $[T_1,T_2]\subseteq[t_1,t_2]$，满足如下条件

$$\int_{T_1}^{T_2}\left[R_\mu(t,a^\nu){}_{t_1}^{C}D_t^\alpha a^\mu - B(t,a^\nu)\right]dt = \int_{\bar{T}_1}^{\bar{T}_2}\left[R_\mu^1(\bar{t},\bar{a}^\nu){}_{\bar{t}_1}^{C}D_{\bar{t}}^\alpha \bar{a}^\mu - B^1(\bar{t},\bar{a}^\nu)\right]d\bar{t} \tag{46}$$

则称分数阶 Pfaff 作用量 (式 (33)) 在无限小变换 (式 (23)) 下准不变的，变换 (式 (23)) 为系统的 Noether 准对称变换.

由式 (46) 确定的 R_μ^1，B^1 和 R_μ，B 具有相同的运动微分方程，此时有

$$\int_{T_1}^{T_2}\left[R_\mu(t,a^\nu){}_{t_1}^{C}D_t^\alpha a^\mu - B(t,a^\nu)\right]dt = \int_{\bar{T}_1}^{\bar{T}_2}\left[R_\mu(\bar{t},\bar{a}^\nu){}_{\bar{t}_1}^{C}D_{\bar{t}}^\alpha \bar{a}^\mu - B(\bar{t},\bar{a}^\nu) + \frac{d}{d\bar{t}}\Delta G(\bar{t},\bar{a}^\nu)\right]d\bar{t} \tag{47}$$

于是有

判据 4 对于 Caputo 导数下分数阶 Birkhoff 系统，如果存在规范函数 $G(t,a^\nu)$ 使得无限小变换 (式 (23)) 的生成元 ξ_0，ξ_μ 满足条件

$$\left(\frac{\partial R_\mu}{\partial t}{}_{t_1}^{C}D_t^\alpha a^\mu - \frac{\partial B}{\partial t}\right)\xi_0 + \left(\frac{\partial R_\mu}{\partial a^\nu}{}_{t_1}^{C}D_t^\alpha a^\mu - \frac{\partial B}{\partial a^\nu}\right)\xi_\nu + \left(R_\mu {}_{t_1}^{C}D_t^\alpha a^\mu - B\right)\dot{\xi}_0 + R_\mu\left[{}_{t_1}^{C}D_t^\alpha\xi_\mu + \xi_0 {}_{t_1}^{C}D_t^\alpha \dot{a}^\mu - {}_{t_1}^{C}D_t^\alpha(\dot{a}^\mu\xi_0)\right] + \frac{d}{dt}G = 0 \tag{48}$$

则变换 (式 (23)) 是系统在一般无限小变换下的 Noether 准对称变换.

证明 由式 (43) 可得出

$$\int_{T_1}^{T_2}\left[R_\mu(t,a^\nu){}_{t_1}^{C}D_t^\alpha a^\mu - B(t,a^\nu)\right]dt = \int_{T_1}^{T_2}\left[R_\mu(\bar{t},\bar{a}^\nu)\frac{d\bar{t}}{dt}{}_{t_1}^{C}D_t^\alpha \bar{a}^\mu - B(\bar{t},\bar{a}^\nu)\frac{d\bar{t}}{dt} + \frac{d}{dt}\Delta G(\bar{t},\bar{a}^\nu)\right]dt \tag{49}$$

由积分区间 $[T_1,T_2]$ 的任意性，式 (49) 等价于

$$R_\mu(t,a^\nu){}_{t_1}^{C}D_t^\alpha a^\mu - B(t,a^\nu) = R_\mu(\bar{t},\bar{a}^\nu)\frac{d\bar{t}}{dt}{}_{t_1}^{C}D_t^\alpha \bar{a}^\mu - B(\bar{t},\bar{a}^\nu)\frac{d\bar{t}}{dt} + \frac{d}{dt}\Delta G(\bar{t},\bar{a}^\nu) \tag{50}$$

注意到

$$\begin{aligned}
&{}_{t_1}^{\mathrm{C}}D_t^{\alpha}\bar{a}^{\mu}(\bar{t}) = \\
&\frac{1}{\Gamma(1-\alpha)}\int_{\bar{t}_1}^{\bar{t}}(\bar{t}-\bar{\tau})^{-\alpha}\frac{\mathrm{d}}{\mathrm{d}\bar{\tau}}\bar{a}^{\mu}(\bar{\tau})\,\mathrm{d}\bar{\tau} = \\
&\frac{1}{\Gamma(1-\alpha)}\int_{t_1}^{t}(t+\Delta t-\tau-\Delta\tau)^{-\alpha}\frac{\mathrm{d}}{\mathrm{d}\tau+\mathrm{d}\Delta\tau}\cdot \\
&(a^{\mu}(\tau)+\Delta a^{\mu}(\tau))\left(1+\frac{\mathrm{d}}{\mathrm{d}\tau}\Delta\tau\right)\mathrm{d}\tau = \\
&\frac{1}{\Gamma(1-\alpha)}\int_{t_1}^{t}(t-\tau)^{-\alpha}\left(\frac{\mathrm{d}}{\mathrm{d}\tau}a^{\mu}(\tau)+\right. \\
&\left.\frac{\mathrm{d}}{\mathrm{d}\tau}\Delta a^{\mu}(\tau)-\alpha\frac{\Delta t-\Delta\tau}{t-\tau}\frac{\mathrm{d}}{\mathrm{d}\tau}a^{\mu}(\tau)\right)\mathrm{d}\tau = \\
&{}_{t_1}^{\mathrm{C}}D_t^{\alpha}a^{\mu}(t)+{}_{t_1}^{\mathrm{C}}D_t^{\alpha}\Delta a^{\mu}(t)-\frac{\alpha}{\Gamma(1-\alpha)}\cdot \\
&\int_{t_1}^{t}(t-\tau)^{-\alpha}\frac{\Delta t-\Delta\tau}{t-\tau}\frac{\mathrm{d}}{\mathrm{d}\tau}a^{\mu}(\tau)\,\mathrm{d}\tau
\end{aligned}\tag{51}$$

其中 $\Delta t=\varepsilon\xi_0(t,a^{\nu}(t))$，$\Delta a^{\mu}(t)=\varepsilon\xi_{\mu}(t,a^{\nu}(t))$．由于

$$\begin{aligned}
&\frac{\alpha}{\Gamma(1-\alpha)}\int_{t_1}^{t}(t-\tau)^{-\alpha}\frac{\Delta t-\Delta\tau}{t-\tau}\frac{\mathrm{d}}{\mathrm{d}\tau}a^{\mu}(\tau)\,\mathrm{d}\tau = \\
&\frac{1}{\Gamma(1-\alpha)}\int_{t_1}^{t}(\Delta t-\Delta\tau)\frac{\mathrm{d}}{\mathrm{d}\tau}a^{\mu}(\tau)\,\mathrm{d}(t-\tau)^{-\alpha} = \\
&\left[\frac{\dot{a}^{\mu}(\tau)}{\Gamma(1-\alpha)}(t-\tau)^{-\alpha}(\Delta t-\Delta\tau)\right]\Bigg|_{t_1}^{t} - \\
&\frac{\Delta t}{\Gamma(1-\alpha)}\int_{t_1}^{t}(t-\tau)^{-\alpha}\frac{\mathrm{d}}{\mathrm{d}\tau}\dot{a}^{\mu}(\tau)\,\mathrm{d}\tau + \\
&\frac{1}{\Gamma(1-\alpha)}\int_{t_1}^{t}(t-\tau)^{-\alpha}\frac{\mathrm{d}}{\mathrm{d}\tau}(\dot{a}^{\mu}(\tau)\Delta\tau)\,\mathrm{d}\tau = \\
&-\Delta t\,{}_{t_1}^{\mathrm{C}}D_t^{\alpha}\dot{a}^{\mu}(t)+{}_{t_1}^{\mathrm{C}}D_t^{\alpha}(\dot{a}^{\mu}(t)\Delta t)
\end{aligned}\tag{52}$$

将式 (52) 代入式 (51)，有

$$\begin{aligned}
&{}_{t_1}^{\mathrm{C}}D_t^{\alpha}\bar{a}^{\mu}(\bar{t}) = {}_{t_1}^{\mathrm{C}}D_t^{\alpha}a^{\mu}(t)+{}_{t_1}^{\mathrm{C}}D_t^{\alpha}\Delta a^{\mu}(t)+ \\
&\Delta t\,{}_{t_1}^{\mathrm{C}}D_t^{\alpha}\dot{a}^{\mu}(t)-{}_{t_1}^{\mathrm{C}}D_t^{\alpha}(\dot{a}^{\mu}(t)\Delta t) = \\
&{}_{t_1}^{\mathrm{C}}D_t^{\alpha}a^{\mu}+\varepsilon\left[{}_{t_1}^{\mathrm{C}}D_t^{\alpha}\xi_{\mu}+\xi_0\,{}_{t_1}^{\mathrm{C}}D_t^{\alpha}\dot{a}^{\mu}-{}_{t_1}^{\mathrm{C}}D_t^{\alpha}(\dot{a}^{\mu}\xi_0)\right]
\end{aligned}\tag{53}$$

将方程 (50) 等号两边对 ε 求导，利用式 (53)，并令 $\varepsilon=0$，得

$$\begin{aligned}
0 = &\left(\frac{\partial R_{\mu}}{\partial t}\xi_0+\frac{\partial R_{\mu}}{\partial a^{\nu}}\xi_{\nu}\right){}_{t_1}^{\mathrm{C}}D_t^{\alpha}a^{\mu}+ \\
&R_{\mu}\left[{}_{t_1}^{\mathrm{C}}D_t^{\alpha}\xi_{\mu}+\dot{\xi}_0\,{}_{t_1}^{\mathrm{C}}D_t^{\alpha}a^{\mu}+\xi_0\,{}_{t_1}^{\mathrm{C}}D_t^{\alpha}\dot{a}^{\mu}-\right. \\
&\left.{}_{t_1}^{\mathrm{C}}D_t^{\alpha}(\dot{a}^{\mu}\xi_0)\right]- \\
&\frac{\partial B}{\partial t}\xi_0-\frac{\partial B}{\partial a^{\nu}}\xi_{\nu}-B\dot{\xi}_0+\frac{\mathrm{d}}{\mathrm{d}t}G
\end{aligned}$$

证毕．

定理 4　对于 Caputo 导数下分数阶 Birkhoff 系统，如果一般无限小变换 (式 (23)) 是系统的 Noether 准对称变换，则

$$\begin{aligned}
I_N = &R_{\mu}\xi_{\mu}+(1-\alpha)R_{\mu}\xi_0\,{}_{t_1}^{\mathrm{C}}D_t^{\alpha}a^{\mu}- \\
&B\xi_0+{}_{t_1}D_t^{1-\alpha}G
\end{aligned}\tag{54}$$

是系统在定义 3 意义下的一个分数阶守恒量．

证明　引进李普希茨变换

$$[t_1,t_2]\ni t\mapsto\sigma f(\lambda)\in[\sigma_1,\sigma_2]\tag{55}$$

当 $\lambda=0$ 时，满足 $t'_{\sigma}=\dfrac{\mathrm{d}t(\sigma)}{\mathrm{d}\sigma}=f(\lambda)=1$．记

$$\begin{aligned}
&L\left(t,a^{\nu}(t),{}_{t_1}^{\mathrm{C}}D_t^{\alpha}a^{\nu}(t)\right) = \\
&\quad R_{\mu}(t,a^{\nu}(t))\,{}_{t_1}^{\mathrm{C}}D_t^{\alpha}a^{\mu}(t)-B(t,a^{\nu}(t))
\end{aligned}\tag{56}$$

则分数阶 Pfaff 作用量 (式 (33)) 成为

$$\begin{aligned}
&S[a^{\nu}(\cdot)] = \int_{t_1}^{t_2}L\left(t,a^{\nu}(t),{}_{t_1}^{\mathrm{C}}D_t^{\alpha}a^{\nu}(t)\right)\mathrm{d}t = \\
&\int_{t_1}^{t_2}\left[R_{\mu}(t,a^{\nu}(t))\,{}_{t_1}^{\mathrm{C}}D_t^{\alpha}a^{\mu}(t)-B(t,a^{\nu}(t))\right]\mathrm{d}t = \\
&\int_{\sigma_1}^{\sigma_2}\left[R_{\mu}(t(\sigma),a^{\nu}(t(\sigma)))\,{}_{\sigma_1}^{\mathrm{C}}D_{t(\sigma)}^{\alpha}a^{\mu}(t(\sigma))-\right. \\
&\left.B(t(\sigma),a^{\nu}(t(\sigma)))\right]t'_{\sigma}\mathrm{d}\sigma
\end{aligned}\tag{57}$$

其中 $t(\sigma_1)=t_1$，$t(\sigma_2)=t_2$，以及

$$\begin{aligned}
&{}_{\sigma_1}^{\mathrm{C}}D_{t(\sigma)}^{\alpha}a^{\mu}(t(\sigma)) = \\
&\frac{1}{\Gamma(1-\alpha)}\int_{t_1/f(\lambda)}^{\sigma f(\lambda)}(\sigma f(\lambda)-\tau)^{-\alpha}\frac{\mathrm{d}}{\mathrm{d}\tau}a^{\mu}\left(\tau f^{-1}(\lambda)\right)\mathrm{d}\tau = \\
&\frac{(t'_{\sigma})^{-\alpha}}{\Gamma(1-\alpha)}\int_{t_1/(t'_{\sigma})^2}^{\sigma}(\sigma-s)^{-\alpha}\frac{\mathrm{d}}{\mathrm{d}s}a^{\mu}(s)\mathrm{d}s = \\
&(t'_{\sigma})^{-\alpha}\,{}_{t_1/(t'_{\sigma})^2}^{\mathrm{C}}D_{\sigma}^{\alpha}a^{\mu}(\sigma)
\end{aligned}\tag{58}$$

将式 (58) 代入式 (57)，得

$$\begin{aligned}
&S[a^{\nu}(\cdot)] = \\
&\int_{\sigma_1}^{\sigma_2}\left[R_{\mu}(t(\sigma),a^{\nu}(t(\sigma)))(t'_{\sigma})^{-\alpha}\,{}_{t_1/(t'_{\sigma})^2}^{\mathrm{C}}D_{\sigma}^{\alpha}a^{\mu}(\sigma)-\right. \\
&\left.B(t(\sigma),a^{\nu}(t(\sigma)))\right]t'_{\sigma}\mathrm{d}\sigma = \\
&\int_{\sigma_1}^{\sigma_2}\bar{L}\left(t(\sigma),a^{\nu}(t(\sigma)),t'_{\sigma},{}_{t_1/(t'_{\sigma})^2}^{\mathrm{C}}D_{\sigma}^{\alpha}a^{\mu}(\sigma)\right)\mathrm{d}\sigma = \\
&\bar{S}[t(\cdot),a^{\nu}(t(\cdot))]
\end{aligned}\tag{59}$$

因此，如果分数阶作用量 $S[a^{\nu}(\cdot)]$ 在定义 5 意义下是准不变的，则分数阶作用量 $\bar{S}[t(\cdot), a^{\nu}(t(\cdot))]$ 在定义 4 意义下是准不变的．由定理 3，得到

$$I_N = \frac{\partial \bar{L}}{\partial t'_{\sigma}}\xi_0 + \frac{\partial \bar{L}}{\partial_{t_1/(t'_{\sigma})^2}{}^{C}D^{\alpha}_{\sigma}a^{\mu}(\sigma)}\xi_{\mu} + {}_{t_1}D^{1-\alpha}_{t}G \tag{60}$$

是系统在定义 3 意义下的分数阶守恒量．当 $\lambda = 0$ 时，有

$${}_{t_1/(t'_{\sigma})^2}{}^{C}D^{\alpha}_{\sigma}a^{\mu}(\sigma) = {}_{t_1}{}^{C}D^{\alpha}_{t}a^{\mu}(t) \tag{61}$$

因此，我们有

$$\frac{\partial \bar{L}}{\partial_{t_1/(t'_{\sigma})^2}{}^{C}D^{\alpha}_{\sigma}a^{\mu}(\sigma)} = R_{\mu} \tag{62}$$

而

$$\begin{aligned}
\frac{\partial \bar{L}}{\partial t'_{\sigma}} = {} & R_{\mu}(t(\sigma), a^{\nu}(t(\sigma)))\frac{\partial}{\partial t'_{\sigma}}\left[\frac{(t'_{\sigma})^{-\alpha}}{\Gamma(1-\alpha)}\int_{t_1/(t'_{\sigma})^2}^{\sigma}(\sigma - s)^{-\alpha}\frac{\mathrm{d}}{\mathrm{d}s}a^{\mu}(s)\mathrm{d}s\right]t'_{\sigma} + \\
& R_{\mu}(t(\sigma), a^{\nu}(t(\sigma)))(t'_{\sigma})^{-\alpha}{}_{t_1/(t'_{\sigma})^2}{}^{C}D^{\alpha}_{\sigma}a^{\mu}(\sigma) - B(t(\sigma), a^{\nu}(t(\sigma))) = \\
& R_{\mu}(t(\sigma), a^{\nu}(t(\sigma)))\frac{-\alpha(t'_{\sigma})^{-\alpha}}{\Gamma(1-\alpha)}\int_{t_1/(t'_{\sigma})^2}^{\sigma}(\sigma - s)^{-\alpha}\frac{\mathrm{d}}{\mathrm{d}s}a^{\mu}(s)\mathrm{d}s + \\
& R_{\mu}(t(\sigma), a^{\nu}(t(\sigma)))(t'_{\sigma})^{-\alpha}{}_{t_1/(t'_{\sigma})^2}{}^{C}D^{\alpha}_{\sigma}a^{\mu}(\sigma) - B(t(\sigma), a^{\nu}(t(\sigma))) = \\
& (1-\alpha)R_{\mu}{}_{t_1}^{C}D^{\alpha}_{t}a^{\mu} - B
\end{aligned} \tag{63}$$

将式 (63) 和式 (62) 代入式 (60)，得到分数阶守恒量 (式 (54))．证毕．

定理 3 和定理 4 可称为 Caputo 导数下分数阶 Birkhoff 系统的 Noether 定理．定理基于 Frederico 和 Torres 提出的分数阶守恒量概念，建立了 Caputo 导数下分数阶 Birkhoff 系统的 Noether 准对称性与 Frederico-Torres 分数阶守恒量之间的联系．如果 $G \equiv 0$，则定理给出了分数阶 Birkhoff 系统的 Noether 对称性与分数阶守恒量之间的联系；如果 $\alpha \to 1$，则定理 3 和定理 4 退化为定理 1 和定理 2，给出了经典 Birkhoff 系统的 Noether 对称性与经典 Noether 守恒量之间的联系．

4 算 例

考虑分数阶 Hojman-Urrutia 问题 [35]，该问题可表示为一个四阶分数阶 Birkhoff 系统，在 Caputo 导数下其 Pfaff 作用量为

$$\begin{aligned}
S[a^{\mu}(\cdot)] = \int_{t_1}^{t_2}\Big\{ & \left(a^2 + a^3\right){}_{t_1}^{C}D^{\alpha}_{t}a^1 + a^4{}_{t_1}^{C}D^{\alpha}_{t}a^3 + \\
& \frac{1}{2}\left[\left(a^3\right)^2 + 2a^2a^3 - \left(a^4\right)^2\right]\Big\}\mathrm{d}t
\end{aligned} \tag{64}$$

试研究该系统的分数阶 Noether 对称性与分数阶守恒量．

分数阶 Birkhoff 方程 (37) 给出

$$\left.\begin{aligned}
& {}_{t}D^{\alpha}_{t_2}a^2 + {}_{t}D^{\alpha}_{t_2}a^3 = 0,\ {}_{t_1}^{C}D^{\alpha}_{t}a^1 - a^3 = 0 \\
& {}_{t_1}^{C}D^{\alpha}_{t}a^1 - a^3 - a^2 + {}_{t}D^{\alpha}_{t_2}a^4 = 0,\ {}_{t_1}^{C}D^{\alpha}_{t}a^3 + a^4 = 0
\end{aligned}\right\} \tag{65}$$

判据方程 (48) 给出

$$\begin{aligned}
& \xi_2\left({}_{t_1}^{C}D^{\alpha}_{t}a^1 - a^3\right) + \xi_3\left({}_{t_1}^{C}D^{\alpha}_{t}a^1 - a^3 - a^2\right) + \\
& \quad \xi_4\left({}_{t_1}^{C}D^{\alpha}_{t}a^3 + a^4\right) + \\
& \quad \Big[\left(a^2 + a^3\right){}_{t_1}^{C}D^{\alpha}_{t}a^1 + a^4{}_{t_1}^{C}D^{\alpha}_{t}a^3 - \\
& \quad \frac{1}{2}\left(a^3\right)^2 - a^2a^3 + \frac{1}{2}\left(a^4\right)^2\Big]\dot{\xi}_0 + \\
& \quad \left(a^2 + a^3\right)\left[{}_{t_1}^{C}D^{\alpha}_{t}\xi_1 + \xi_0{}_{t_1}^{C}D^{\alpha}_{t}\dot{a}^1 - {}_{t_1}^{C}D^{\alpha}_{t}\left(\dot{a}^1\xi_0\right)\right] + \\
& \quad a^4\left[{}_{t_1}^{C}D^{\alpha}_{t}\xi_3 + \xi_0{}_{t_1}^{C}D^{\alpha}_{t}\dot{a}^3 - {}_{t_1}^{C}D^{\alpha}_{t}\left(\dot{a}^3\xi_0\right)\right] + \dot{G} = 0
\end{aligned} \tag{66}$$

方程 (66) 有解

$$\xi_0^1 = 1,\ \xi_1^1 = 1,\ \xi_2^1 = \xi_3^1 = \xi_4^1 = 0,\ G^1 = 0 \tag{67}$$

$$\xi_0^2 = -1,\ \xi_1^2 = \xi_2^2 = \xi_3^2 = \xi_4^2 = 0,\ G^2 = c^2 \tag{68}$$

式 (67) 对应系统的 Noether 对称变换，式 (68) 对应系统的 Noether 准对称变换．根据定理 4，得到

$$\begin{aligned}
I_N^1 = {} & a^2 + a^3 + (1-\alpha)\Big[\left(a^2 + a^3\right){}_{t_1}^{C}D^{\alpha}_{t}a^1 + \\
& a^4{}_{t_1}^{C}D^{\alpha}_{t}a^3\Big] - \frac{1}{2}\left[\left(a^3\right)^2 + 2a^2a^3 - \left(a^4\right)^2\right]
\end{aligned} \tag{69}$$

$$I_N^2 = -(1-\alpha)\left[\left(a^2+a^3\right){}_{t_1}^{\ C}D_t^{\alpha}a^1 + a^4{}_{t_1}^{\ C}D_t^{\alpha}a^3\right] + \frac{1}{2}\left[\left(a^3\right)^2 + 2a^2a^3 - \left(a^4\right)^2\right] + {}_{t_1}D_t^{1-\alpha}c^2 \tag{70}$$

其中 c^2 为任意常数. 式 (69) 是由 Noether 对称性 (式 (67)) 导致的分数阶守恒量，式 (70) 是由 Noether 准对称性 (式 (68)) 导致的分数阶守恒量. 当 $\alpha \to 1$ 时，式 (69) 和式 (70) 成为

$$I_N^1 = a^2 + a^3 - \frac{1}{2}\left[\left(a^3\right)^2 + 2a^2a^3 - \left(a^4\right)^2\right] \tag{71}$$

$$I_N^2 = \frac{1}{2}\left[\left(a^3\right)^2 + 2a^2a^3 - \left(a^4\right)^2\right] + c^2 \tag{72}$$

式 (71) 和式 (72) 是经典 Hojman-Urrutia 问题的 Noether 守恒量.

5 结 论

由于应用分数阶微积分可以更准确地描述和研究复杂系统的动力学行为和物理过程，同时 Birkhoff 系统动力学是 Hamilton 力学的推广，因此研究分数阶 Birkhoff 系统动力学具有重要意义. 本工作一是建立了经典 Birkhoff 系统和 Caputo 导数下分数阶 Birkhoff 系统的 Noether 准对称性的定义和判据；二是基于时间重新参数化方法证明了经典 Birkhoff 系统和 Caputo 导数下分数阶 Birkhoff 系统的 Noether 定理，该定理建立了系统的 Noether 准对称性与分数阶守恒量之间的内在联系. 以往关于 Caputo 导数下分数阶 Birkhoff 系统基于 Frederico-Torres 分数阶守恒量定义的 Noether 定理以及经典 Birkhoff 系统的 Noether 定理都是本文之特例. 由于 Caputo 分数阶导数定义解决了 Riemann-Liouville 定义中的分数阶初值问题，从而在工程和实际问题的动力学建模中得到了更广泛的应用，因此本文的方法和结果可望得到广泛应用和进一步发展. 最后必须指出，基于 Frederico-Torres 分数阶守恒量定义如何建立 Noether 准对称性与守恒量的关系尚属开放的课题，例如 Riemann-Liouville 导数下分数阶 Noether 定理的推广等.

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