The Price Model of Common Derivative Securities

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Abstract　Considered the Random Differential Equation that state variables based on, the paper thinks that at least n+1 kinds of the price of exchangeable securities depend on PDE，its matrix and corresponding pricing express that some or all state variables meet.

Key words　derivative securities, PDE, Ito

1 Introduction

When we discuss the Price Model of Derivative Securities, commonly we study the n state variables $\theta_i\left(1 \le i \le n\right)$, wich follow the continuous time $It\hat{o}$ extended model.[1]

$$d\theta_i = m_i\theta_i dt + s_i\theta_i dW_i$$

We use the standard non-arbitrage method to give the Partial Differential Equation (PDE) that the price of the derivative securities $f\left(\cdot\right)$ satisfies, which only depends on the state variable $\theta_i\left(1 \le i \le n\right)$ and the time t,

$$\frac{\partial f}{\partial t} + \sum_{i=1}^{m}\theta_i\frac{\partial f}{\partial\theta_i}\left(m_i - \lambda_i s_i\right) + \frac{1}{2}\sum_{i,k=1}^{m}\rho_{ik}s_i s_k\theta_i\theta_k\frac{\partial^2 f}{\partial\theta_i\partial\theta_k} = rf$$

Among them，dW_i is the Wiener Process, parameter m_i and s_i are the expectation increasing ratio and fluctuating ratio of θ_i . r is the instant non-risk interest rate, ρ_{ik} is the correlation coefficient of dW_i and dW_k .

According to the PDE, after adding some definite boundary conditions, we can get the given derivative security.[1]

In this discussion, the state variable θ_i which follows the $It\hat{o}$ process depends on dW_i. Let's start from the Random Differential Equation that the state variable follows, consider at least n securities, which prices rely on the PDE and their matrixes that some or all n state variables meet.

2 Models

Let the state variable $\theta^i\left(1 \le i \le n\right)$ follow the Random Differential Equation here:

$$\begin{cases} d\theta_t^1 = b^1(t,\theta_t)dt + \sum_{j=1}^{n}\sigma_{1j}(t,\theta_t)dW_t^j \\ \vdots \\ d\theta_t^{n1} = b^n(t,\theta_t)dt + \sum_{j=1}^{n}\sigma_{nj}(t,\theta_t)dW_t^j \end{cases} \qquad (1)$$

Among them,

$$\theta_t = (\theta_t^1,\cdots,\theta_t^n)^T, \sigma_{ij}(t,\theta_t) = \mathrm{cov}(W_t^i, W_t^j)$$

We write

$$b_t = (b^1(t,\theta_t),\cdots,b^n(t,\theta_t))^T = (b_t^1,\cdots,b_t^n)^T$$

$$W_t = (W^1(t),\cdots,W^n(t))^T, \quad \Sigma = \begin{bmatrix} \sigma_{11} & \cdots & \sigma_{1n} \\ \vdots & \ddots & \vdots \\ \sigma_{n1} & \cdots & \sigma_{nn} \end{bmatrix} = (\sigma_{ij}(t,\theta))_{n\times n}$$

Define

$$dW=(dW_t^{\ 1},\cdots,dW_t^{\ n})^T,d\theta(t)=(d\theta_t^{\ 1},\cdots,d\theta_t^{\ n})^T \quad ,$$

Then（1）can be expressed to

$$d\theta_t=b_t dt+\sum dW_t \qquad (1)$$

Let $f(\cdot)\in C^2$ be any price of a exchangeable security, it depends on the state variable θ^j, according to the $It\hat{o}$ Theory, f follows the processes as below.

$$df=\frac{\partial f}{\partial t}dt+\sum_{i=1}^{n}\frac{\partial f}{\partial\theta^i}d\theta^i+1/2\sum_{i,j=1}^{n}\frac{\partial^2 f}{\partial\theta^i\partial\theta^j}d\left\langle d\theta^i,d\theta^j\right\rangle$$

$$=\frac{\partial f}{\partial t}dt+\sum_{i=1}^{n}\frac{\partial f}{\partial\theta^i}[b_t^i dt+\sum_{j=1}^{n}\sigma_{ij}dW^j]+1/2\sum_{i,j=1}^{n}(\frac{\partial^2 f}{\partial\theta^i\partial\theta^j}\sum_{k=1}^{n}\sigma_{ik}\sigma_{jk})dt$$

$$=[\frac{\partial f}{\partial t}+\sum_{i=1}^{n}\frac{\partial f}{\partial\theta^i}b^i+1/2\sum_{i,j=1}^{n}(\frac{\partial^2 f}{\partial\theta^i\partial\theta^j}\sum_{k=1}^{n}\sigma_{ik}\sigma_{jk})]dt+\sum_{i,j=1}^{n}\frac{\partial f}{\partial\theta^i}\sigma_{ij}dW_t^{\ j}$$

We write

$$L=\frac{1}{2}\sum_{i,j=1}^{n}a_{ij}(t,\theta_t)\frac{\partial^2}{\partial\theta^i\partial\theta^j}+\sum_{i=1}^{n}b^i(t,\theta_t)\frac{\partial}{\partial\theta^i}$$

Among them,

$$a_{ij}(t,\theta_t)=\sum_{k=1}^{n}\sigma_{ik}(t,\theta_t)\sigma_{jk}(t,\theta_t)$$

$$g_j(t,\theta_t)=\sum_{i=1}^{n}\frac{\partial f}{\partial\theta^i}\sigma_{ij}$$

Then

$$df=(\frac{\partial f}{\partial t}+Lf)dt+\sum_{j=1}^{n}g_j dW_t^{\ j} \qquad (2)$$

Because n exchangeable securities all depend on θ^i . We use f_j to express j prices of the exchangeable securities, then we can get the equation group:

$$df_j=(\frac{\partial f_j}{\partial t}+Lf_j)dt+\sum_{i=1}^{n}g_{ij}dW_t^{\ j} \qquad (3)$$

$$\triangleq\mu_j dt+\sum_{i=1}^{n}g_{ij}dW^j$$

Among them, $\mu_j=\dfrac{\partial f_j}{\partial t}+Lf_j$

We write

$$\begin{bmatrix} \frac{1}{f_1}\frac{\partial f_1}{\partial \theta^1} & \cdots & \frac{1}{f_1}\frac{\partial f_1}{\partial \theta^n} \\ \vdots & & \vdots \\ \frac{1}{f_n}\frac{\partial f_n}{\partial \theta^1} & \cdots & \frac{1}{f_n}\frac{\partial f_n}{\partial \theta^n} \end{bmatrix} = \begin{bmatrix} \frac{\partial \ln f_1}{\partial \theta^1} & \cdots & \frac{\partial \ln f_1}{\partial \theta^n} \\ \vdots & & \vdots \\ \frac{\partial \ln f_n}{\partial \theta^1} & \cdots & \frac{\partial \ln f_n}{\partial \theta^n} \end{bmatrix} \overset{\Delta}{=} \Delta$$

$$(\frac{1}{f_1}\mu_1, \cdots, \frac{1}{f_n}\mu_n)^T \overset{\Delta}{=} \overline{\mu}$$

$$(\frac{df_1}{f_1}, \cdots, \frac{df_n}{f_n})^T = (d\ln f_1, \cdots, d\ln f_n)^T \overset{\Delta}{=} d\ln F$$

Among them, $F = (f_1, \cdots, f_n)^T$ .Then the equation group（3）can be expressed in matrix,

$$d\ln F = \overline{\mu}dt + \Delta \sum dW_t \qquad (3)$$

3 Expressing The PDE With Matrix

In the equation group (3), there are n exchangeable securities and n Wiener processes. Using these securities, we can construct a instant non-risk portfolio π . Let χ_i be the investment proportion on f_i , we write

$$\pi = \sum_{i=1}^{n} x_i f_i \qquad \sum_{i=1}^{n} x_i = 1$$

Its yield is

$$\frac{d\pi}{\pi} = \sum_{i=1}^{n} x_i \frac{df_i}{f_i} = \sum_{i=1}^{n} x_i d\ln f_i = x^T d\ln F$$

$$= \sum_{i=1}^{n} x_i \overline{\mu}_i dt + \sum_{i=1}^{n} x_i (\sum_{i=1}^{n}\sum_{k=1}^{n} \Delta_{ik} \sum_{kj}) dW^j$$

$$= x^T \overline{\mu}dt + x^T \Delta \sum dW_t$$

Theory 1 if $|\Delta| \neq 0$, then linear equation group

$$\begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} = \begin{bmatrix} 1 & 1 & \cdots & 1 \\ & & & \\ & \Delta\Sigma & & \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \qquad (4)$$

can have nonzero solution.

Using the solution, we can construct the portfolio π , it can include no noises, and the income of the portfolio π is：

$$\frac{d\pi}{\pi} = \sum_{i=1}^{n} x_i \overline{\mu}_i dt = x^T \overline{\mu} dt$$

If there are no arbitrary opportunities, the income of the portfolio must be the non-risk interest r , we write

$$d\pi / \pi = rdt$$

Thus

$$\begin{cases} \sum_{i=1}^{n} x_i \, \bar{\mu}_i = r \\ \sum_{j=1}^{n} x_j \sum_{k=1}^{n} \Delta_{ik} \, \Sigma_{kj} = 0 \end{cases} \Leftrightarrow \begin{cases} \sum_{i=1}^{n} x_i (\bar{\mu}_i - r) = 0 \\ \sum_{j=1}^{n} x_j \sum_{j,k=1}^{n} \Delta_{ik} \, \Sigma_{kj} = 0 \end{cases} \quad (5)$$

(5) can be expressed in matrix

$$\begin{cases} X^T (\bar{\mu} - r1) = 0 \\ \Delta \Sigma X^T = 0 \end{cases} \quad (5)$$

of them, $1 = (1, \cdots, 1)^T$

Theory 2 If $|\Delta| \neq 0$, then there exist $\lambda_1, \cdots, \lambda_n$, that

$$\bar{\mu}_i - r = \sum_{j=1}^{n} \lambda_j \sum_{k=1}^{n} \Delta_{ik} \, \Sigma_{kj} \quad (6)$$

Prove: We write δ_j to express the j line of $\Delta\Sigma$, then $n+1$ vector groups of n dimensions $\sigma_{1,} \cdots, \sigma_n, \bar{\mu} - r1$ are linear correlative. Because $|\Delta| \neq 0$, we know that $\bar{\mu} - r1$ is the linear combination of $\sigma_{1,} \cdots, \sigma_n$, and $\bar{\mu} - r1$ can be linearly expressed by $\sigma_{1,} \cdots, \sigma_n$. We write the expressed coefficient $\lambda_1, \cdots, \lambda_n$, then (6) come into existence.

We write $\Lambda = (\lambda_{1,} \cdots, \lambda_n)^T$, (6) can be expressed in matrix

$$\bar{\mu} - r1 = \Delta \Sigma \Lambda \quad (7)$$

we put the defining equation of $\bar{\mu}$ into (7),

$$\left(\frac{\mu_1}{f_1} \cdots \frac{\mu_n}{f_n} \right)^T - r1 = \Delta \Sigma \Lambda$$

We write

$$\tilde{F}^{-1} = diag[1/f_1 \cdots 1/f_n] \qquad \tilde{F} = diag[f_1 \cdots f_n]$$

The equation above can be expressed to

$$\mu^T \tilde{F}^{-1} - r1 = \Delta \Sigma \Lambda \quad (8)$$

We write it into ponderance form,

$$\frac{\mu_l}{f_l} - r = \sum_{j=1}^{n} \left(\sum_{j,k=1}^{n} \Delta_{lk} \, \Sigma_{kj} \right) \lambda_j$$

Put μ_1 into the above equation

$$\frac{1}{f_t} \left[\frac{\partial f_t}{\partial t} + L f_t \right] - r = \sum_{j=1}^{n} \sum_{j,k=1}^{n} \Delta_{lk} \, \Sigma_{kj} \lambda_j$$

The equation goes as

$$\frac{1}{f_l} \left[\frac{\partial f_l}{\partial t} + \sum_{i=1}^{n} b_i \frac{\partial f_l}{\partial \theta^i} + \frac{1}{2} \sum_{i,j=1}^{n} a_{ij} \frac{\partial^2 f_l}{\partial \theta^i \partial \theta^j} \right] - r = \sum_{j,k=1}^{n} \Delta_{lk} \, \Sigma_{kj} \lambda_j$$

That is

$$\frac{1}{f_l}\left[\frac{\partial f_l}{\partial t}+\sum_{i=1}^{n}b_i\frac{\partial f_l}{\partial\theta^i}+\frac{1}{2}\sum_{i,j=1}^{n}\frac{\partial^2 f_l}{\partial\theta^i\partial\theta^j}\sum_{m=1}^{n}\sigma_{im}\sigma_{mj}\right]-r=\sum_{j,k=1}^{n}\frac{1}{f_l}\frac{\partial f_l}{\partial\theta^k}\sigma_{kj}\lambda_j$$

It can be evolved to

$$\frac{\partial f_l}{\partial t}+\sum_{i=1}^{n}\left(b_i-\sum_{j=1}^{n}\sigma_{ij}\lambda_j\right)\frac{\partial f_l}{\partial\theta^i}+\frac{1}{2}\sum_{i,j=1}^{n}\left(\sum_{m=1}^{n}\sigma_{im}\sigma_{jm}\right)\frac{\partial^2 f_l}{\partial\theta^i\partial\theta^j}=rf_l \qquad (9)$$

It can be expressed in matrix

$$\frac{\partial F}{\partial t}+\left(b-\Sigma\Lambda\right)\frac{\partial F}{\partial\theta}+\frac{1}{2}I^T\,\Sigma\left(\nabla^2 F\right)\Sigma=rF \qquad (10)$$

Thus we have

Theory 3 n exchangeable securities that depends on the state variables and the time t satisfies the PDE (10), among them,

$$\nabla^2 F=\left(\frac{\partial^2 f_l}{\partial\theta^i\partial\theta^j}\right)$$

We can see from the equation (9) that the PDE that any derivative security depended on n object securities satisfies is similar to (9.10) in structure, only changing the increasing ratio of each object securities' θ_i from b_i to $\left(b_i-\sum_{j=1}^{n}\sigma_{ij}\lambda_j\right)$, it demonstrates that if we change b_i to $\left(b_i-\sum_{j=1}^{n}\sigma_{ij}\lambda_j\right)$ while the fluctuation ratio is unchangeable and maintain the original covariance of θ_i,θ_k, the price equation of the derivative securities can still be used, and still be the non-risk interest discount of expecting profit. Then as to the equation

$$d\theta=\left(b-\Sigma\Lambda\right)dt+\Sigma\,dW$$

It can define any price u_i of exchangeable security of the derivative security F, thus

$$u_i=\hat{\mathrm{E}}\left[e^{-\int_t^T b_i(s)ds}\Psi_i\left(\theta_T^i\right)\middle|\theta_T^i=\theta\right]$$

and it's the solution in the boundary condition of $V_i\left(T,\theta\right)=\Psi_i\left(\theta\right)$

Among them, $\hat{\mathrm{E}}$ is the expectation of changing the expected increasing ratio of b_i to the $\left(b_i-\sum_{j=1}^{n}\sigma_{ij}\lambda_j\right)$.

Reference

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