Error Estimation of a Kind of Rational Spline

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Abstract

This paper deals with the approximation properties of a kind of rational spline with linear denominator when the function being interpolated is C^3 in an interpolating interval. Error estimate expressions of interpolating functions are derived, convergence is established, the optimal error coefficient, c_i, is proved to be symmetric about the parameters of the rational interpolation and it is bounded. Finally, the precise jump measurements of the second derivatives of the interpolating function at the knots are given.

Keywords: rational spline; error analysis; approximation.

1 Introduction

Spline interpolation is a useful and powerful tool in computer aided geometric design. Polynomial splines are the kinds of splines which are widely applied in curve and surface design [1-3,9-11,14,16,17,21]. Since the interpolating function is unique for the given interpolation data, local modification of the interpolating curve or surface is impossible under the conditions that the interpolating data are not changed. In recent years, the rational spline with parameters has received attention in the literature [4-8,12,13,15,18-20]. For the given interpolating data, the change of the parameters causes the change of the interpolating curves or surfaces, so that the interpolating curves or surfaces may be modified to be the shape needed if suitable parameters exist. That is, the uniqueness of the interpolating function for the given data is replaced by the uniqueness of the interpolating curve or surface for the given data and the selected parameters.

There are rational cubic splines with linear, quadratic or cubic denominator [7,9,13]. Since there are parameters in the interpolations, those splines are effectively used in the design and modification of curves, such as region control and convexity control [6,7,13,15,18,19]. Also, because of the parameters, the approximation properties are difficult to study. Some results are given in [5] for simple cases. When the function being interpolated, $f(t)$, has continuous second-order derivative, the error estimations of those interpolations were derived in [5], and it was shown that, from the point of view of the magnitude of the optimal error coefficient, the spline with linear denominator has better

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approximation to the function being interpolated than the rational interpolation with quadratic or cubic denominator. A further question is that, when the function being interpolated is smoother than C^2, such as C^3, how about the approximation of the interpolating function? And furthermore, the interpolating function $P(t) \in C^1$ for the parameters in the non-constrained case, what is the jump measurement of $P''(t)$ at the knots? Those questions will be answered in this paper.

The paper will deal with the approximation in the case that the function being interpolated, $f(t)$, has continuous third-order derivative in the interpolating interval. For this, the rational cubic spline with linear denominator will be restated briefly in section 2. Section 3 studies the error estimation of the interpolating functions, gives the optimal error coefficient c_i which depends on the parameters in the interpolating function, and proves that c_i is symmetric about the parameters. Section 4 is about the jump measurement of the second derivatives of the interpolating function at the knots.

2 Rational spline with linear denominator

A rational cubic spline with linear denominator was given in [7]. Let $t_0 < t_1 < \cdots < t_n$ be the knot spacing and $\{f_i, d_i, i = 0, 1, \dots n\}$ be a given set of data points, where f_i, d_i are, respectively, the function values and the derivative values at the knots of the function being interpolated, $f(t)$. Define the C^1-continuous, piecewise rational interpolating function by

$$P(t)|_{[t_i,t_{i+1}]} = \frac{p_i(t)}{q_i(t)}, \tag{1}$$

where

$$\begin{aligned} p_i(t) &= (1-\theta)^3\alpha_i f_i + \theta(1-\theta)^2 V_i + \theta^2(1-\theta)W_i + \theta^3\beta_i f_{i+1}, \\ q_i(t) &= (1-\theta)\alpha_i + \theta\beta_i, \\ \theta &= (t-t_i)/h_i, \\ h_i &= t_{i+1} - t_i, \end{aligned}$$

and

$$V_i = (2\alpha_i + \beta_i)f_i + \alpha_i h_i d_i, \quad W_i = (\alpha_i + 2\beta_i)f_{i+1} - \beta_i h_i d_{i+1}, \tag{2}$$

with $\alpha_i, \beta_i > 0$. The function $P(t)$ satisfies $P(t_i) = f_i, \quad P'(t_i) = d_i, \quad i = 0, 1, \cdots, n.$

Obviously, when $\alpha_i = \beta_i$, the interpolation defined by (1) is the standard cubic Hermite interpolation.

3 Error estimation of the interpolation

This section deals with the error estimation of the interpolation when the function being interpolated $f(t) \in C^3[t_0, t_n]$. Since the interpolation is local, without loss of generality it is necessary only to consider the error in the subinterval $[t_i, t_{i+1}]$. It is easy to show that the interpolation is exactly held for the polynomial function being interpolated in which the degree is no more than two. Let $P(t)$ be the rational interpolating function of $f(t)$ in $[t_i, t_{i+1}]$ defined by (1). Using the Peano-Kernel Theorem [21] gives the following

$$R[f] = f(t) - P(t) = \frac{1}{2}\int_{t_i}^{t_{i+1}} f^{(3)}(\tau) R_t[(t-\tau)_+^2]\, d\tau, \tag{3}$$

where

$$\begin{aligned} R_t[(t-\tau)_+^2] &= \begin{cases} (t-\tau)^2 - \frac{1}{(1-\theta)\alpha_i+\beta_i\theta}\{\theta^2(1-\theta)[(2\beta_i+\alpha_i)(t_{i+1}-\tau)^2 - 2\beta_i h_i(t_{i+1}-\tau)] \\ \qquad +\theta^3\beta_i(t_{i+1}-\tau)^2\}, \qquad t_i < \tau < t; \\ -\frac{1}{(1-\theta)\alpha_i+\beta_i\theta}\{\theta^2(1-\theta)[(2\beta_i+\alpha_i)(t_{i+1}-\tau)^2 - 2\beta_i h_i(t_{i+1}-\tau)] \\ \qquad +\theta^3\beta_i(t_{i+1}-\tau)^2\}, \qquad t < \tau < t_{i+1}. \end{cases} \\ &= \begin{cases} r(\tau,t), & t_i < \tau < t; \\ s(\tau,t), & t < \tau < t_{i+1}. \end{cases} \end{aligned}$$

The function $R_t[(t-\tau)_+^2]$ is called the kernel of the integral (3). To derive the error estimate representation, $|R[f]|$, the properties of the kernel functions $r(\tau,t)$ and $s(\tau,t)$ need to be studied first, and then the values $\int_{t_i}^{t} |r(\tau,t)| \mathrm{d}\tau$ and $\int_{t}^{t_{i+1}} |s(\tau,t)| \mathrm{d}\tau$ will be calculated. To stating clearly, the proof process includes two parts: Part 1 is about the calculation of the value $\int_{t_i}^{t} |r(\tau,t)| \mathrm{d}\tau$, Part 2 is about the calculation of the value $\int_{t}^{t_{i+1}} |s(\tau,t)| \mathrm{d}\tau$. Combining Part 1 and Part 2 will complete the proof.

Part 1. Study the properties of the function $r(\tau,t)$. Consider $r(\tau,t), \tau \in [t_i,t]$ as a function of τ, $r(\tau,t)$ is a quadratic polynomial of variable τ. According to the construction of the kernel functions $r(\tau,t)$ in using the Peano-Kernel Theorem, for all $\theta \in [0,1]$

$$r(t_i,t) = 0.$$

By simple computation,

$$r(t,t) = \frac{\theta^2(1-\theta)^2((\alpha_i+\beta_i)\theta-\alpha_i)h_i^2}{(1-\theta)\alpha_i+\theta\beta_i}.$$

Let

$$(\alpha_i+\beta_i)\theta - \alpha_i = 0$$

be considered as an equation in θ; its root in (0,1) is

$$\theta^* = \frac{\alpha_i}{\alpha_i+\beta_i}. \tag{4}$$

It is easy to show that, when $\theta \le \theta^*, r(t,t) \le 0$ and when $\theta \ge \theta^*, r(t,t) \ge 0$. To see the sign of $r(\tau,t)$ in $[t_i,t]$, rewrite $r(\tau,t)$ as

$$\begin{aligned} r(\tau,t) &= \frac{1}{(1-\theta)\alpha_i+\theta\beta_i}[((1-\theta)^2(1+\theta)\alpha_i + \theta(1-\theta)^2\beta_i)(t-\tau)^2 \\ &\quad -2\theta^2(1-\theta)^2 h_i(\alpha_i+\beta_i)(t-\tau) + \theta^2(1-\theta)^2 h_i^2((\alpha_i+\beta_i)\theta-\alpha_i)], \end{aligned}$$

then it can be found that the second root of $r(\tau,t)$ is

$$\tau^* = t - \frac{h_i\theta((\theta-1)\alpha_i+\theta\beta_i)}{(1+\theta)\alpha_i+\theta\beta_i}$$

besides the root t_i, and when $\theta > \theta^*, t_i < \tau^* < t$, when $\theta < \theta^*, \tau^* > t$. Thus, when $\theta < \theta^*, r(\tau,t) < 0$ for all $\tau \in [t_i, t]$, so

$$\int_{t_i}^{t} |r(\tau,t)|\mathrm{d}\tau = \int_{t_i}^{t}(-r(\tau,t))\mathrm{d}\tau = \frac{\theta^3(1-\theta)^2((2-\theta)\alpha_i-\theta\beta_i)h_i^3}{3((1-\theta)\alpha_i+\beta_i\theta)}. \tag{5}$$

When $\theta > \theta^*$, the values of $r(\tau,t)$ varies from negative to positive on the two sides of τ^*, so

$$\begin{aligned}&\int_{t_i}^{t} |r(\tau,t)|\mathrm{d}\tau = \int_{t_i}^{\tau^*}(-r(\tau,t))\mathrm{d}\tau + \int_{\tau^*}^{t} r(\tau,t)\mathrm{d}\tau\\ =\ & \frac{\theta^3(1-\theta)^2[((2-\theta)\alpha_i-\theta\beta_i)((1+\theta)\alpha_i+\theta\beta_i)^2+2((2+\theta)\alpha_i+\theta\beta_i)((\theta-1)\alpha_i+\theta\beta_i)^2]h_i^3}{3((1-\theta)\alpha_i+\beta_i\theta)((1+\theta)\alpha_i+\theta\beta_i)^2}.\end{aligned} \tag{6}$$

Part 2. Study the properties of the function $s(\tau,t)$. Consider $s(\tau,t), \tau \in [t, t_{i+1}]$ as a function of τ. Similar as discussed for $r(\tau,t)$,

$$s(t_{i+1},t) = 0$$

and

$$s(t,t) = r(t,t),$$

and by a similar analysis as in part 1, one can see that when $\theta \le \theta^*, s(t,t) \le 0$ and when $\theta \ge \theta^*, s(t,t) \ge 0$. Rewriting $s(\tau,t)$ as

$$s(\tau,t) = \frac{-(t_{i+1}-\tau)}{(1-\theta)\alpha_i+\theta\beta_i}[(\theta^2(1-\theta)\alpha_i+\theta^2(2-\theta)\beta_i)(t_{i+1}-\tau)-2\theta^2(1-\theta)\beta_i h_i],$$

and denoting

$$\tau_* = t_{i+1} - \frac{2\theta^2(1-\theta)\beta_i h_i}{\theta^2(1-\theta)\alpha_i+\theta^2(2-\theta)\beta_i},$$

it is easy to show that when $\theta \le \theta^*, s(\tau,t)$ varies from negative to positive on the two sides of τ_*, and when $\theta \ge \theta^*, s(\tau,t)$ remains positive in (t, t_{i+1}), where θ^* is defined by (4). Thus, when $\theta \le \theta^*$

$$\begin{aligned}&\int_{t}^{t_{i+1}} |s(\tau,t)|\mathrm{d}\tau = \int_{t}^{\tau_*}(-s(\tau,t))\mathrm{d}\tau + \int_{\tau_*}^{t_{i+1}} s(\tau,t)\mathrm{d}\tau\\ =\ & \frac{\theta^2(1-\theta)^3[(1-\theta)^3(\alpha_i+\beta_i)^3+3(\theta-1)\alpha_i\beta_i^2+3((1+\theta)\beta_i^3)]h_i^3}{3((1-\theta)\alpha_i+\beta_i\theta)((1-\theta)\alpha_i+(2-\theta)\beta_i)^2},\end{aligned} \tag{7}$$

and when $\theta \ge \theta^*$

$$\int_{t}^{t_{i+1}} |s(\tau,t)|\mathrm{d}\tau = \frac{\theta^2(1-\theta)^3((1+\theta)\beta_i-(1-\theta)\alpha_i)h_i^3}{3((1-\theta)\alpha_i+\beta_i\theta)}. \tag{8}$$

Thus, combining (5) and (7), it can be shown that, when $\theta \le \theta^*$,

$$|f(t)-P(t)| \le \frac{\|f^{(3)}\|}{2}\int_{t_i}^{t_{i+1}} |R_t[(t-\tau)_+^2]|\mathrm{d}\tau = \|f^{(3)}\|h_i^3 w_1(\alpha_i,\beta_i,\theta),$$

where

$$w_1(\alpha_i,\beta_i,\theta) = \frac{\theta^2(1-\theta)^2[(1-\theta)^2\alpha_i^3+(1-\theta)(3-\theta)\alpha_i^2\beta_i+\theta(2-\theta)\alpha_i\beta_i^2+(4-4\theta-\theta^2)\beta^3]}{6((1-\theta)\alpha_i+\beta_i\theta)((1-\theta)\alpha_i+(2-\theta)\beta_i)^2} \tag{9}$$

and when $\theta \ge \theta^*$, combining (6) and (8) gives

$$|f(t)-P(t)| \le \frac{\|f^{(3)}\|}{2}\int_{t_i}^{t_{i+1}} |R_t[(t-\tau)_+^2]|\mathrm{d}\tau = \|f^{(3)}\|h_i^3 w_2(\alpha_i,\beta_i,\theta),$$

where

$$w_2(\alpha_i,\beta_i,\theta) = \frac{\theta^2(1-\theta)^2[(-\theta^2+6\theta-1)\alpha_i^3+(1-\theta^2)\alpha_i^2\beta_i+\theta(2+\theta)\alpha_i\beta_i^2+\theta^2\beta^3]}{6((1-\theta)\alpha_i+\beta_i\theta)((1+\theta)\alpha_i+\theta\beta_i)^2}. \tag{10}$$

Based on the analysis above, the theorem on the error estimation of the interpolating function is obtained as follows

THEOREM 3.1 *For $f(t) \in C^3[t_0,t_n]$, let $P(t)$ be the rational interpolating function of $f(t)$ in $[t_i,t_{i+1}]$ defined by (1). For the positive parameters α_i and β_i, the error of the interpolating function $P(t)$ satisfies*

$$|f(t)-P(t)| \quad \le \quad \|f^{(3)}(t)\|h_i^3 c_i$$

with

$$c_i = \max_{0\le\theta\le 1} w(\alpha_i,\beta_i,\theta), \tag{11}$$

$$w(\alpha_i,\beta_i,\theta) = \begin{cases} w_1(\alpha_i,\beta_i,\theta), & 0\le\theta\le\theta^*; \\ w_2(\alpha_i,\beta_i,\theta), & \theta_*\le\theta\le 1; \end{cases}$$

where, $w_1(\alpha_i,\beta_i,\theta)$ and $w_2(\alpha_i,\beta_i,\theta)$ are defined by (9) and (10), respectively.

In the special case, let $\alpha_i = \beta_i$, then the interpolation defined by (1) is the standard cubic Hermite interpolation. In this case the functions $w_1(\alpha_i,\beta_i,\theta)$ and $w_2(\alpha_i,\beta_i,\theta)$ become

$$w_1(\theta) = 4\theta^2(1-\theta)^3/(3(3-2\theta)^2), \quad 0\le\theta\le\frac{1}{2};$$
$$w_2(\theta) = 4\theta^3(1-\theta)^2/(3(1+2\theta)^2), \quad \frac{1}{2}\le\theta\le 1$$

respectively. Since

$$\max\{\max_{0\le\theta\le\frac{1}{2}} w_1(\theta), \max_{\frac{1}{2}\le\theta\le 1} w_2(\theta)\} = 1/96$$

it follows that the error coefficient

$$c_i = 1/96, \tag{12}$$

which is the well-known result for the standard cubic Hermite interpolation.

By the definitions given for $w_1(\alpha_i, \beta_i, \theta)$ and $w_2(\alpha_i, \beta_i, \theta)$, it can easily be shown that

$$w_1(\alpha_i, \beta_i, \theta) = w_2(\beta_i, \alpha_i, 1-\theta),$$

Thus, there is the following theorem

THEOREM 3.2 *The optimal error constant c_i in Theorem 3.1 is symmetric about the parameters α_i and β_i, namely*

$$\max_{0\le\theta\le1} w_1(\alpha_i, \beta_i, \theta) = \max_{0\le\theta\le1} w_2(\beta_i, \alpha_i, 1-\theta). \tag{13}$$

Since $w_1(\alpha_i, \beta_i, \theta)$ and $w_2(\alpha_i, \beta_i, \theta)$ are continuous functions of the variate θ in the interval $[0, 1]$, so the coefficient c_i is bounded. In fact, there is the following boundary theorem about the optimal error constant c_i.

THEOREM 3.3 *For any given positive parameters α_i and β_i, the error optimal constants c_i in Theorem 3.1 are bounded with*

$$\frac{1}{96} \le c_i \le \frac{2}{81}.$$

Proof From Theorem 3.1 and Theorem 3.2, if

$$\frac{1}{96} \le \max_{0\le\theta\le1} w_1(\alpha_i, \beta_i, \theta) \le \frac{2}{81}. \tag{14}$$

then Theorem 3.3 holds.

Let $\alpha_i = \lambda_i\beta_i$, then (9) becomes

$$w_1(\alpha_i, \beta_i, \theta) = \frac{\theta^2(1-\theta)^2[(1-\theta)^2\lambda_i^3 + (1-\theta)(3-\theta)\lambda_i^2 + \theta(2-\theta)\lambda_i + (4-4\theta-\theta^2)]}{6((1-\theta)\lambda_i+\theta)((1-\theta)\lambda_i+(2-\theta))^2}. \tag{15}$$

Denote

$$w^*(\lambda_i, \theta) = \frac{[(1-\theta)^2\lambda_i^3 + (1-\theta)(3-\theta)\lambda_i^2 + \theta(2-\theta)\lambda_i + (4-4\theta-\theta^2)]}{((1-\theta)\lambda_i+\theta)((1-\theta)\lambda_i+(2-\theta))^2}, \tag{16}$$

then

$$\frac{\mathrm{d}w^*(\lambda_i,\theta)}{\mathrm{d}\lambda_i} = \frac{\lambda_i^3(1-\theta)^3 + \lambda_i^2(1-\theta)^2(6-3\theta) - \lambda_i(1-\theta)(21\theta^2-36\theta+12) - (9\theta^3-6\theta^2-12\theta+8)}{((1-\theta)\lambda_i+\theta)^2((1-\theta)\lambda_i+(2-\theta))^3},$$

and

$$\frac{\mathrm{d}w^*(\lambda_i,\theta)}{\mathrm{d}\lambda_i}=0$$

gives

$$\lambda_i^3(1-\theta)^3+\lambda_i^2(1-\theta)^2(6-3\theta)-\lambda_i(1-\theta)(21\theta^2-36\theta+12)-(9\theta^3-6\theta^2-12\theta+8)=0. \quad (17)$$

Denote $\delta=\lambda_i(1-\theta)$, then (17) becomes

$$\delta^3+\delta^2(6-3\theta)-\delta(21\theta^2-36\theta+12)-(9\theta^3-6\theta^2-12\theta+8)=0. \quad (18)$$

Consider (18) as the equation in the variable δ, $\delta\in(0,+\infty)$; the roots of (18) are

$$\delta_1=2-3\theta,\quad \delta_2=(2\sqrt{3}+3)\theta-(2\sqrt{3}+4),\quad \delta_3=(3-2\sqrt{3})\theta+(2\sqrt{3}-4).$$

Since $\delta=\lambda_i(1-\theta)>0$, and for any $\theta\in[0,1]$, $\delta_2<0$ and $\delta_3<0$, so the roots δ_2 and δ_3 are omitted. Thus, for the relative fixed $\theta\in[0,1]$, equation (18) has only one root $\delta_1=2-3\theta$ in $(0,+\infty)$, then

$$\lambda_i=\frac{2-3\theta}{1-\theta} \quad (19)$$

is the only critical point of (16) in $(0,+\infty)$. Substitute (19) into (15), the right side of (15) can be simplified to

$$w_1(\alpha_i,\beta_i,\theta)=\frac{\theta^2(3-4\theta)}{24},$$

and it is easy to find that

$$\max_{0\le\theta\le1} w_1(\alpha_i,\beta_i,\theta)=\max_{0\le\theta\le1}\frac{\theta^2(3-4\theta)}{24}=\frac{1}{96}. \quad (20)$$

On the other hand, consider the two cases $\lambda_i\to+\infty$ and $\lambda_i\to0$. First, consider $\lambda_i\to+\infty$: from (16) and (15),

$$\lim_{\lambda_i\to+\infty} w_1(\alpha_i,\beta_i,\theta)=\frac{\theta^2(1-\theta)^4}{6(1-\theta)^3}=\frac{\theta^2(1-\theta)}{6}, \quad (21)$$

and it is easy to show that

$$\max_{0\le\theta\le1}\frac{\theta^2(1-\theta)}{6}=\frac{2}{81}. \quad (22)$$

Then, consider $\lambda_i\to0$: from (16) and (15),

$$\lim_{\lambda_i\to0} w_1(\alpha_i,\beta_i,\theta)=\frac{\theta(1-\theta)^2(4-4\theta-\theta^2)}{6(\theta-2)^2}. \quad (23)$$

Denote the right side of (23) by $g(\theta)$, i.e.

$$g(\theta) = \frac{\theta(1-\theta)^2(4-4\theta-\theta^2)}{6(\theta-2)^2},$$

then

$$\frac{\mathrm{d}g(\theta)}{\mathrm{d}\theta} = \frac{(1-\theta)(3\theta^4-3\theta^3-30\theta^2+36\theta-8)}{6(\theta-2)^3}.$$

Let

$$\frac{\mathrm{d}g(\theta)}{\mathrm{d}\theta} = 0,$$

then

$$\theta = 1,$$

or

$$3\theta^4 - 3\theta^3 - 30\theta^2 + 36\theta - 8 = 0. \tag{24}$$

The roots of equation (24) are

$$\theta_1 = \frac{1-\sqrt{33}}{4} + \frac{1}{12}\sqrt{450+30\sqrt{33}}, \quad \theta_2 = \frac{1-\sqrt{33}}{4} - \frac{1}{12}\sqrt{450+30\sqrt{33}},$$
$$\theta_3 = \frac{1+\sqrt{33}}{4} + \frac{1}{12}\sqrt{450-30\sqrt{33}}, \quad \theta_4 = \frac{1+\sqrt{33}}{4} - \frac{1}{12}\sqrt{450-30\sqrt{33}},$$

and only θ_1 and θ_4 are in (0,1). Thus, when $\lambda_i \to 0$, it can be shown that

$$\max_{0\le\theta\le1} w_1(\alpha_i,\beta_i,\theta) = \max_{0\le\theta\le1} = \frac{\theta(1-\theta)^2(4-4\theta-\theta^2)}{6(\theta-2)^2} = 0.0246913\cdots. \tag{25}$$

Combining (20),(22) and (25), completes the proof .

From the definition of the functions $w_1(\alpha_i,\beta_i,\theta)$ and $w_2(\alpha_i,\beta_i,\theta)$, it is easy to find the optimal error constant c_i by (11). Table 1 gives some c_i for the given parameters α_i and β_i, and shows the symmetric property.

Table 1. Values of c_i for some values of parameters α_i, β_i .

i	α_i	β_i	θ	c_i
1	100.0	100.0	0.5000	0.0104
2	1.0	1.0	0.5000	0.0104
3	100.0	1.0	0.6600	0.0240
4	1.0	100.0	0.3400	0.0240
5	10.0	5.0	0.5510	0.0116
6	5.0	10.0	0.4490	0.0116
7	8.0	6.0	0.5230	0.0160
8	6.0	8.0	0.4770	0.0160
9	1.1	0.9	0.5160	0.0105
10	0.9	1.1	0.4840	0.0105
11	0.3	0.2	0.5320	0.0108
12	0.2	0.3	0.4680	0.0108

From Theorem 3.3, the convergence theorem is obtained as follows

THEOREM 3.4 *If $f(t) \in C^3[t_0, t_n]$ and $P(t)$ is the rational interpolant function of $f(t)$ in $[t_0, t_n]$ defined by (1), for the positive parameters $\alpha_i, \beta_i, i = 0, 1, 2, \cdots, n-1$, $P(t)$ converges to $f(t)$ in $[t_0, t_n]$, namely*

$$\lim_{h \to 0} P(t) = f(t),$$

where, $h = \max_i h_i$

4 Jump in the second derivatives

From the definition of the interpolation, the interpolating function $P(t) \in C^1[t_0, t_n]$, so the second derivative $P''(t)$ has a jump at the knots. For the jump measurement, the following theorem can be proved.

THEOREM 4.1 *Let $f(t) \in C^3[t_0, t_n]$ and let $P(t)$ be the rational interpolating function of $f(t)$ in $[t_0, t_n]$ defined by (1). If the knots are equally spaced, namely, $h = \frac{t_n - t_0}{n}$, for the positive parameters α_i and β_i, the jump measurement of the second derivative at the knot t_i satisfies*

$$|P''(t_i+) - P''(t_i-)| \leq \|f^{(3)}(t)\| h \bar{c}_i,$$

where

$$\bar{c}_i = W(\alpha_{i-1}, \beta_{i-1}, \alpha_i, \beta_i)$$

and

$$W(\alpha_{i-1}, \beta_{i-1}, \alpha_i, \beta_i) = \frac{4\alpha_{i-1}^3 + 3\alpha_{i-1}\beta_{i-1}^2 + \beta_{i-1}^3}{3(\beta_{i-1}(2\alpha_{i-1} + \beta_{i-1})^2)} + \frac{\alpha_i^3 + 3\alpha_i^2\beta_i + 4\beta_i^3}{3(\alpha_i(\alpha_i + 2\beta_i)^2)}.$$

Proof. Since

$$P''(t) = (h_i^2((1-\theta)\alpha_i+\theta\beta_i)^3)^{-1}\cdot Q(\theta),$$

where

$$\begin{aligned} Q(\theta) = {} & ((1-\theta)\alpha_i+\theta\beta_i)^2(6(1-\theta)\alpha_i f_i+(6\theta-4)V_i+(2-6\theta)W_i \\ & +6\theta\beta_i f_{i+1})-2(\beta_i-\alpha_i)((1-\theta)\alpha_i+\theta\beta_i)(-3(1-\theta)^2\alpha_i f_i \\ & +(1-4\theta+3\theta^2)V_i+(2\theta-3\theta^2)W_i+3\theta^2\beta_i f_{i+1})+2(\beta_i-\alpha_i)^2\cdot \\ & ((1-\theta)^3\alpha_i f_i+\theta(1-\theta)^2V_i+\theta^2(1-\theta)W_i+\theta^3\beta_i f_{i+1}), \end{aligned}$$

then

$$\begin{aligned} P''(t_i+)-P''(t_i-) = {} & \frac{2}{h^2}[-(1+\frac{2\alpha_{i-1}}{\beta_{i-1}}))f_{i-1}+(\frac{2\alpha_{i-1}}{\beta_{i-1}}-\frac{2\beta_i}{\alpha_i})f_i+(1+\frac{2\beta_i}{\alpha_i})f_{i+1}] \\ & -\frac{2}{h}[\frac{\alpha_{i-1}}{\beta_{i-1}}d_{i-1}+(2+\frac{\alpha_{i-1}}{\beta_{i-1}}+\frac{\beta_i}{\alpha_i})d_i+\frac{\beta_i}{\alpha_i}d_{i+1}]. \end{aligned} \tag{26}$$

When $f(t)\in C^3[t_0,t_n]$, for any $i\in\{1,2,\cdots,n-1\}$, denote $r_i(f)=P''(t_i+)-P''(t_i-)$. Using the Peano-Kernel Theorem gives

$$r_i(f)=P''(t_i+)-P''(t_i-)=\frac{1}{2!}\int_{t_{i-1}}^{t_{i+1}} f^{(3)}(\tau)r_t[(t-\tau)_+^2]\,\mathrm{d}\tau, \tag{27}$$

where

$$\begin{aligned} r_t[(t-\tau)_+^2] &= \begin{cases} \frac{2}{h^2}((\frac{2\alpha_{i-1}}{\beta_{i-1}}-\frac{2\beta_i}{\alpha_i})(t_i-\tau)^2+(1+\frac{2\beta_i}{\alpha_i})(t_{i+1}-\tau)^2) \\ -\frac{2}{h}(2(2+\frac{\alpha_{i-1}}{\beta_{i-1}}+\frac{\beta_i}{\alpha_i})(t_i-\tau)+\frac{2\beta_i}{\alpha_i}(t_{i+1}-\tau)), & t_{i-1}<\tau<t_i; \\ \frac{2}{h^2}(1+\frac{2\beta_i}{\alpha_i})(t_{i+1}-\tau)^2-\frac{4\beta_i}{h\alpha_i}(t_{i+1}-\tau), & t_i<\tau<t_{i+1}; \end{cases} \\ &= \begin{cases} \frac{2}{h^2}(1+\frac{2\alpha_{i-1}}{\beta_{i-1}})(t_i-\tau)^2-\frac{4}{h}(1+\frac{\alpha_{i-1}}{\beta_{i-1}})(t_i-\tau)+2, & t_{i-1}<\tau<t_i; \\ \frac{2}{h^2}(1+\frac{2\beta_i}{\alpha_i})(t_{i+1}-\tau)^2-\frac{4\beta_i}{h\alpha_i}(t_{i+1}-\tau), & t_i<\tau<t_{i+1}; \end{cases} \\ &= \begin{cases} m(\tau), & t_{i-1}<\tau<t_i; \\ n(\tau), & t_i<\tau<t_{i+1}. \end{cases} \end{aligned}$$

Consider

$$\frac{2}{h^2}(1+\frac{2\alpha_{i-1}}{\beta_{i-1}})(t_i-\tau)^2-\frac{4}{h}(1+\frac{\alpha_{i-1}}{\beta_{i-1}})(t_i-\tau)+2=0 \tag{28}$$

as a quadratic equation in τ. The root of (28) in (t_{i-1},t_i) is

$$t^*=t_i-\frac{h\beta_{i-1}}{2\alpha_{i-1}+\beta_{i-1}}.$$

Thus, when $t_{i-1} \le \tau \le t^*, m(\tau) \le 0$ and $t^* \le \tau \le t_i, m(\tau) \ge 0$, so

$$\int_{t_{i-1}}^{t_i} |m(\tau)| \mathrm{d}\tau = \int_{t_{i-1}}^{t^*} (-m(\tau)) \mathrm{d}\tau + \int_{t^*}^{t_i} m(\tau) \mathrm{d}\tau = \frac{2(4\alpha_{i-1}^3 + 3\alpha_{i-1}\beta_{i-1}^2 + \beta_{i-1}^3)h}{3\beta_{i-1}(2\alpha_{i-1} + \beta_{i-1})^2}. \tag{29}$$

Similarly, $n(\tau) = 0$ has a root t_* in (t_i, t_{i+1}) and

$$t_* = t_{i+1} - \frac{2\beta_i h}{\alpha_i + 2\beta_i}.$$

It is easy to show that when $t_i \le \tau \le t_*, n(\tau) \ge 0$ and $t_* \le \tau \le t_{i+1}, n(\tau) \le 0$, so

$$\int_{t_i}^{t_{i+1}} |n(\tau)| \mathrm{d}\tau = \int_{t_i}^{t_*} n(\tau) \mathrm{d}\tau + \int_{t_*}^{t_{i+1}} (-n(\tau)) \mathrm{d}\tau = \frac{2(\alpha_i^3 + 3\alpha_i^2\beta_i + 4\beta_i^3)h}{3\alpha_i(\alpha_i + 2\beta_i)^2}. \tag{30}$$

Combining (27),(29) and (30), it can be derived that

$$|P''(t_i+) - P''(t_i-)| \le \frac{1}{2!}\|f^{(3)}\|[\frac{2(4\alpha_{i-1}^3 + 3\alpha_{i-1}\beta_{i-1}^2 + \beta_{i-1}^3)h}{3\beta_{i-1}(2\alpha_{i-1} + \beta_{i-1})^2} + \frac{2(\alpha_i^3 + 3\alpha_i^2\beta_i + 4\beta_i^3)h}{3\alpha_i(\alpha_i + 2\beta_i)^2}]. \tag{31}$$

The proof is complete.

From (31), for the standard cubic Hermite interpolation,

$$|P''(t_i+) - P''(t_i-)| \le \frac{16}{27}\|f^{(3)}\|h. \tag{32}$$

This result was given in [22].

5 Remarks

(1). Some optimal error constants c_i for the given parameters α_i and β_i are given in Table 1. It shows that c_i are not only symmetric about the the parameters α_i and β_i, but change very little, although α_i and β_i vary widely, it is consilient with Theorem 3.3 and Theorem 3.4. It also shows that the interpolation is stable about the parameters.

(2). The standard cubic Hermite interpolation is the special case of the rational spline defined by (1) when $\alpha_i = \beta_i$; (12) and (32) confirm this relationship.

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