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The complete Schwarzschild interior and exterior solution in the harmonic coordinate system

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An exact solution of Schwarzschild interior and exterior solution for a star which has a uniform density and whose center pressure approaches infinite, in the harmonic coordinate system, is given. This solution enables us to demonstrate that in known literatures, a term in the Schwarzschild exterior solution is usually ignored in such a system. Neglect of the term directly leads to contradiction of the continuity of the Schwarzschild interior and exterior solution expressed in the system. © *1998 American Institute of Physics.* [S0022-2488(98)02911-9]

The Harmonic coordinate system (HCS) is an important coordinate system in general relativity. It can be used in studying, e.g., the gravitational fields produced by stable stars,[1–3] black holes,[4,5] gravitational waves,[6] supergravity,[7] quantum gravity,[8] etc. The most fundamental result of HCS is the usual Schwarzschild exterior solution (ES) under HCS.[1] In this paper, we are going to point out that the fundamental ES is not complete indeed if both the Schwarzschild interior solution (IS) and ES are transformed into those in the HCS and then examine their continuity at the boundary.

For simplicity, we will use the rational units in which gravitational constant G and light velocity c are unity.

At first let us examine the general Schwarzschild type solution in the so-called standard coordinate system (SCS)[1] as

$$d\tau^2 = E(r)dt^2 - G(r)dr^2 - r^2(d\theta^2 + \sin^2\theta\, d\phi^2). \tag{1}$$

This solution can be expressed in the following coordinates X_μ ($\mu = 0,1,2,3$):

$$X_0 = t, \tag{2}$$

$$X_1 = R(r)\sin\theta\cos\phi, \tag{3}$$

$$X_2 = R(r)\sin\theta\sin\phi, \tag{4}$$

$$X_3 = R(r)\cos\theta. \tag{5}$$

In these coordinates, the proper time (1) becomes:[1]

$$d\tau^2 = E(r)dt^2 - \frac{r^2}{R^2}d\mathbf{X}^2 - \left(\frac{G}{R^2 R_r'^2} - \frac{r^2}{R^4}\right)(\mathbf{X}\cdot d\mathbf{X}), \tag{6}$$

where $R_r' = dR/dr$.

In the new coordinates, the continuity of the solution at the surface of a spherically symmetrical star of radius a is that the metric components and the so-called extrinsic curvature[9] are continuous at $r = a$, respectively. For metric (6) they are

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$$E_{\rm in}(a)=E_{\rm ex}(a),\quad G_{\rm in}(a)=G_{\rm ex}(a),$$
$$R_{\rm in}(a)=R_{\rm ex}(a),\quad R'_{\rm in}(a)=R'_{\rm ex}(a). \tag{7}$$

where subscripts ''in'' and ''ex'' denote the interior and exterior solution of E, G and R in Eq. (6) respectively.

The coordinates X_μ are to be harmonic if and only if they satisfy the harmonic condition:

$$\Box^2 X_\mu \equiv g^{\lambda\nu}\left[\frac{\partial^2 X_\mu}{\partial x^\nu \partial x^\lambda}-\Gamma^\gamma_{\lambda\nu}\frac{\partial X_\mu}{\partial x^\gamma}\right]=0. \tag{8}$$

It leads to that $R(r)$ must satisfy the differential equation:[1]

$$\frac{d}{dr}\left[r^2\sqrt{\frac{E(r)}{G(r)}}\frac{dR}{dr}\right]-2\sqrt{E(r)G(r)}R=0. \tag{9}$$

Evidently, only after $R(r)$ is solved can we say the metric (6) in HCS is obtained. So, it is necessary to solve ES $R_{\rm ex}(r)$ and IS $R_{\rm in}(r)$, respectively. For ES, $R_{\rm ex}(r)$ is easily available. We have known ES in the SCS for a long time as[1,2,10]

$$E_{\rm ex}(r)=1-\frac{2M}{r}, \tag{10}$$

$$G_{\rm ex}(r)=\frac{1}{1-2M/r}, \tag{11}$$

in which M is a constant of integration. Substituting these relations into Eq. (9), we have

$$\left(1-\frac{2M}{r}\right)R''_{rr}+\frac{2}{r^2}(r-M)R'_r-\frac{2R}{r^2}=0. \tag{12}$$

The general form of the ES $R_{\rm ex}(r)$ is in the interval $0<|r-2M|<\infty$:

$$R_{\rm ex}(r)=C(r-M)+C_1\left[\frac{r-M}{2M}\ln\left(1-\frac{2M}{r}\right)+1\right], \tag{13}$$

where C and C_1 are two arbitrary constants and the constant C can be conveniently chosen to be 1. It is evident in mathematics that the second term cannot be discarded, or say, we cannot put $C_1=0$, when the solution applying to physics objects with $r>2m$. Note that for all spherically symmetrical stable stars, uniform or not, $r\geqslant 9M/4>2M$ always holds.[1] It means that this term probably plays a role for all stars, otherwise the last two equations in continuity equations (7) are not compatible. Surprisingly, in known literature, for example, Refs. 1–5 and 8, C_1 is made zero. In the rest of this paper, an exact solvable star model is given to demonstrate $C_1\neq 0$.

As is well-known, solving IS $R_{\rm in}(r)$ from Eq. (9) is not so simple because the explicit forms of ISs depend on the structure of the stars. For some stars, solving Eq. (9) is mathematically formidable. Fortunately, for a uniform density star whose center pressure approaches infinity, Eq. (9) can be solve analytically. The explicit form of IS for this star will be presented.

First, we need to know the metric components $E_{\rm in}(r), G_{\rm in}(r)$ of IS in the SCS. For an arbitrary uniform density star in the SCS they are in general:[10]

$$E_{\rm in}(r)=\left[A-B\sqrt{1-\frac{r^2}{R_0^2}}\right]^2, \tag{14}$$

$$G_{\rm in}(r)=\frac{1}{1-r^2/R_0^2}, \tag{15}$$

where $R_0^2=3/(8\pi\rho_0)$,[10] A and B are two constants of integration. The above IS can be simplified by considering the internal pressure $p(r)$ of the star, which is[10]

$$8\pi p(r)=\frac{1}{R_0}\left(\frac{3B\xi(r)-A}{A-B\xi(r)}\right), \tag{16}$$

where $\xi(r)=\sqrt{1-r^2/R_0^2}$. Since the pressure equals zero at the surface of the sphere $r=a$, we have

$$3B\xi(a)-A=0, \tag{17}$$

which shows that constants A and B are not independent. And since the star we are interested in has infinite pressure at $r=0$, we have from Eqs. (16) and (17) two relations:

$$A=B,\quad a=\sqrt{\frac{8}{9}}\,R_0. \tag{18}$$

Substituting these relations into Eqs. (14) and (15) we have the simplified forms of $E_{\text{in}}(r)$ and $G_{\text{in}}(r)$ as

$$E_{\text{in}}(r)=A^2\left[1-\sqrt{1-\frac{8r^2}{9a^2}}\right]^2, \tag{19}$$

$$G_{\text{in}}(r)=\frac{1}{1-8r^2/9a^2}. \tag{20}$$

There are still two arbitrary constants A in IS (19) and M in ES [(10) and (11)]. We can fix them by utilizing the first two equations in (7). And they give:

$$M=\frac{4}{3}\pi\rho_0 a^3,\quad A=\frac{1}{2},\quad a=\frac{9M}{4}. \tag{21}$$

So far, both the ES [(10) and (11)] and the IS [(19) and (20)] in the SCS for the special uniform density star are complete.

Second, we need to know the IS $R_{\text{in}}(r)$ in the HCS. From Eq. (9) and the above results the equation determining the IS $R_{\text{in}}(r)$ can be established as

$$R''_{rr}+R'_r\left[\frac{2}{r}+\frac{r}{R_0^2}\frac{1}{[1-\xi(r)]\xi(r)}-\frac{r}{R_0^2}\frac{1}{\xi^2(r)}\right]-\frac{2R}{r^2\xi^2(r)}=0. \tag{22}$$

After lengthy and delicate work, we obtain the following general solution of Eq. (22):

$$R_{\text{in}}(r)=C_2z^{n/2}(1-z)^{m/2}F(\alpha,\beta,\gamma;z)+C_3z^{n/2}(1-z)^{m/2}z^{1-\gamma}F(\alpha-\gamma+1,\beta-\gamma+1,2-\gamma;z), \tag{23}$$

in which

$$n=\frac{-3+\sqrt{17}}{2}, \tag{24}$$

$$m=1, \tag{25}$$

$$\alpha=\frac{m+n}{2}=\frac{-1+\sqrt{17}}{4}, \tag{26}$$

$$\beta=\frac{m+n+6}{2}=\frac{11+\sqrt{17}}{4}, \tag{27}$$

$$\gamma = \frac{2n+5}{2} = \frac{2+\sqrt{17}}{2}, \tag{28}$$

$$z = \tfrac{1}{2}(1-\xi(r)), \tag{29}$$

$$F(\alpha,\beta,\gamma;z) = 1 + \sum_{k=1}^{\infty} \frac{(\alpha)_k(\beta)_k}{(\gamma)_k} \frac{z^k}{k!}, \tag{30}$$

$$(\alpha)_k = \alpha(\alpha+1)(\alpha+2)\cdots(\alpha+k-1), \tag{31}$$

$$C_2 \text{ and } C_3 \text{ are two arbitrary constants.} \tag{32}$$

Thus the general ES $R_{\text{ex}}(r)$ and IS $R_{\text{in}}(r)$ for the special uniform density star are all obtained. Now we are in a position to discuss the constants C_1, C_2, C_3 in Eqs. (13) and (23). Evidently we must set $C_3 = 0$ for $z^{n/2+1-\gamma} \sim r^{-1/4(3+\sqrt{17})} \to \infty$ at the vicinity $r=0$. And importantly, we cannot set $C_1 = 0$ again. Otherwise the mathematical consistency of the last two equations in continuity equations (7) will be violated. In fact the constants C_1 and C_2 are

$$C_1 = \frac{(a-M)g'(a) - g(a)}{g(a)f'(a) - g'(a)f(a)}, \tag{33}$$

$$C_2 = \frac{(a-M)f'(a) - f(a)}{g(a)f'(a) - g'(a)f(a)}, \tag{34}$$

where

$$f(r) = \frac{r-M}{2M} \ln\left(1 - \frac{2M}{r}\right) + 1, \tag{35}$$

$$g(r) = z^{n/2}(1-z)^{m/2} F(\alpha,\beta,\gamma;z). \tag{36}$$

Substituting the results (21) into C_1 and C_2, they are simply:

$$C_1 = 0.060\,498a, \tag{37}$$

$$C_2 = 0.594\,473a. \tag{38}$$

Now the complete forms of the IS $R_{\text{in}}(r)$ and ES $R_{\text{ex}}(r)$ in HCS are obtained as

$$R_{\text{ex}}(r) = (r-M) + 0.060\,498a\left[\frac{r-M}{2M} \ln\left(1 - \frac{2M}{r}\right) + 1\right], \tag{39}$$

$$R_{\text{in}}(r) = 0.594\,473a z^{n/2}(1-z)^{m/2} F(\alpha,\beta,\gamma;z). \tag{40}$$

In other words, from Eqs. (2)–(6), the complete Schwarzschild interior and exterior solution in the HCS for the star is obtained. As we expected, the constant C_1 has a nonvanishing value. The curve showing $R(r)$ versus r is plotted in Fig. 1, in which we can see the curve at point $r=a$ is smoothly connected indeed.

In summary, we can conclude that the usual ES in HCS is not complete in mathematics, or strictly speaking, is not the correct mathematical solution in HCS corresponding to the ES in the SCS, for a term is neglected as $C_1 = 0$. The neglected term reflects how the presence of the mass distribution affects the HCS in the surrounding empty space; it is a purely general-relativistic consequence. However, this does not mean that the usual ES are not a correct solution of the Einstein field equation. In fact they are. In light of the principles in general relativity which permits us to use arbitrary coordinate systems, the usual ES in HCS is an exterior solution in a new coordinate system X_μ (2)–(5) with $R(r) = r - M$. But the new coordinate system is not the

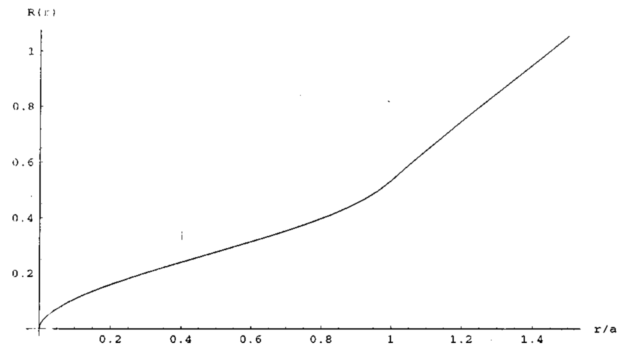

FIG. 1. The curve showing the function relation between $R(r)$ and r in the interval [0,1.5a].

complete HCS at all; and surprisingly, all literature used the HCS in this incomplete sense when an explicit form of HCS was needed.[1–5,8] Finally, we would like to point out that sometimes the neglect of the term may not have a significant influence in physics. Even at the surface of our simplified uniform density star model, only in a percentage point of the results can this term play a role, and the proportion of the second term to the first is approximately 0.04.

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