ELSEVIER

Nuclear Physics B 476 (1996) 548–555

NUCLEAR PHYSICS B

Thermodynamics of stationary axisymmetric Einstein–Maxwell dilaton–axion black hole

Jiliang Jing

CCAST (World Laboratory), P.O. Box 8730, Beijing 100080, China and Institute of Physics and Physics Department [1], Hunan Normal University, Changsha, Hunan 410081, China

Received 21 March 1996; accepted 23 April 1996

Abstract

The thermodynamics of a stationary axisymmetric Einstein–Maxwell dilaton–axion (EMDA) black hole is investigated using general statistical physics methods. It is shown that entropy and energy have the same form as for the Kerr–Newman charged black hole, but temperature, heat capacity and chemical potential have a different form. However, it is shown that the Bardeen–Carter–Hawking laws of black hole thermodynamics are valid for the stationary axisymmetric EMDA black hole. The black hole possesses second-order phase transitions as does the Kerr–Newman black hole because its heat capacity diverges but both the Helmholtz free energy and entropy are continuous at some value of J and Q. Another interesting result is that the action I can be expressed as $I = S + \beta\Omega_{rh} J$ for general stationary black holes in which the external material contribution to mass and angular momentum vanishes. When $J = 0$, i.e., for static black holes, the result reduces to $I = S$.

PACS: 04.60.+n; 97.60.Lf; 04.65.+e; 11.17.+y

1. Introduction

Recently a lot of interest has arosen in obtaining classical dilaton–axion black hole solutions in string theory and investigating their properties [1–4]. In particular attention was focused on the thermodynamics of these black holes. The thermodynamics of a static $U(1)^2$ dilaton black hole [1] was studied by R. Kallosh, T. Ortin and A. Peet [3]. The entropy of a scalar field in the background of a static dilatonic black hole, obtained by D. Garfinkle, G.T. Horowitz and A. Stromiger [1], was investigated by A. Ghosh and

[1] Postal address.

PII S0550-3213(96)00222-2

P. Mitra [4]. The objective of this paper is to investigate thermodynamics of the stationary axisymmetric EMDA black hole [2] and to see how the results differ from the Kerr–Newman black hole as a result of different horizons and singularities as compared to the Kerr–Newman black hole.

In order to study thermodynamics of the stationary axisymmetric EMDA black hole we shall use general statistical physics methods. This new way has the aim of allowing one to consistently apply Feynman's path integral representation [5] of a statistical partition function in the presence of black hole solutions. The approach [6,7] takes explicit account of the significant radial variation of temperature and other thermodynamic quantities when thermal equilibrium exists in the presence of strong inhomogeneous gravitational fields.

The paper is organized as follows: The metric of the stationary axisymmetric EMDA black hole [2] is introduced in Section 2. In Section 3 we work out the Euclidean action for the present model, starting from the gravitational action appropriate canonical boundary. In Section 4 we first construct a statistical partition function and Helmoltz free energy by use of the Euclidean action, and then investigate the thermodynamics of the stationary axisymmetric EMDA black hole. We summarize and discuss our conclusions in the last section.

2. Stationary axisymmetric EMDA black hole

The four-dimensional low-energy Lagrangian obtained from heterotic string theory is [2]

$$S=\frac{1}{16\pi}\int d^4x\,\sqrt{-g}\Big(R-2g^{\mu\nu}\nabla_\mu\Phi\nabla_\nu\Phi-\tfrac{1}{2}e^{4\Phi}g^{\mu\nu}\nabla_\mu K_a\nabla_\nu K_a$$
$$-e^{-2\Phi}g^{\mu\lambda}g^{\nu\rho}F_{\mu\nu}F_{\lambda\rho}-K_aF_{\mu\nu}\tilde{F}^{\mu\nu}\Big), \qquad (2.1)$$

with

$$\tilde{F}_{\mu\nu}=-\tfrac{1}{2}\sqrt{-g}\,\varepsilon_{\mu\nu\alpha\beta}F^{\alpha\beta},$$

where R is the scalar Riemann curvature, $g^{\mu\nu}$ is the metric four-dimensional tensor, Φ is the massless dilaton field, $F_{\mu\nu}$ is the electromagnetic antisymmetric tensor field, and K_a is the axion field dual to the three-index antisymmetric tensor field $H=-\exp(4\Phi)*dK_a/4$.

The stationary axisymmetric EMDA black hole solution (we take the solution $b=0$ in Eq. (14) in Ref. [2]; the reason we use this solution is that the solution $b\neq 0$ cannot be interpreted properly as a black hole) is given by [2]

$$ds^2=-\frac{\Sigma-a^2\sin^2\theta}{\Delta}dt^2$$
$$+\frac{\Delta}{\Sigma}dr^2+\Delta\,d\theta^2+\frac{\sin^2\theta}{\Delta}\Big[(r^2+a^2-2Dr)^2-\Sigma a^2\sin^2\theta\Big]\,d\varphi'^2$$
$$-\frac{2a\sin^2\theta}{\Delta}\big[(r^2+a^2-2Dr)-\Sigma\big]\,dt\,d\varphi', \qquad (2.2)$$

with

$$\Sigma = r^2 - 2mr + a^2, \qquad \Delta = r^2 - 2Dr + a^2 \cos^2\theta \tag{2.3}$$

and

$$e^{2\Phi} = \frac{W}{\Delta} = \frac{\omega}{\Delta}(r^2 + a^2 \cos^2\theta), \qquad \omega = e^{2\Phi_0}, \tag{2.4}$$

$$K_a = K_0 + \frac{2aD\cos\theta}{W}, \tag{2.5}$$

$$A_t = \frac{1}{\Delta}(Qr - ga\cos\theta), \qquad A_r = A_\theta = 0,$$

$$A_{\varphi'} = \frac{1}{a\Delta}\left(-Qra^2\sin^2\theta + g(r^2 + a^2)a\cos\theta\right). \tag{2.6}$$

The mass M, angular momentum J, electric charge Q, and magnetic charge P of the black hole are given by

$$M = m - D, \quad J = a(m - D), \quad Q = \sqrt{2\omega D(D - m)}, \quad P = g. \tag{2.7}$$

The above results show that the stationary axisymmetric EMDA black hole differs considerably from the Kerr–Newman black hole. As a result the stationary axisymmetric EMDA black hole shows several different properties as compared to the Kerr–Newman black hole: (a) Two horizons of the Kerr–Newman black hole are given by $r_\pm = M \pm \sqrt{M^2 - Q^2 - a^2}$, whereas in the case of the stationary axisymmetric EMDA black hole we have $r_\pm = (M - (Q^2/2\omega M)) \pm \sqrt{\left(M - (Q^2/2\omega M)\right)^2 - a^2}$; (b) the Kerr–Newman metric has singularities at $r^2 + \cos^2\theta = 0$, whereas the stationary axisymmetric EMDA black hole has singularities at $r^2 - 2Dr + \cos^2\theta = 0$. It is therefore worthwhile to investigate some new interesting features appearing in thermodynamics of the stationary axisymmetric EMDA black hole in order to see how these differ from the Kerr–Newman black hole.

3. The action

The total action for the stationary axisymmetric EMDA black hole is given by the following volume integral:

$$I = \frac{1}{16\pi}\int_\Sigma (\mathscr{L}_{\text{matter}} - R)\sqrt{g}\,\mathrm{d}x^4 + \frac{1}{8\pi}\int_{\partial\Sigma}(K - K_0)\sqrt{h}\,\mathrm{d}x^3, \tag{3.1}$$

with

$$\mathscr{L}_{\text{matter}} = 2g^{\mu\nu}\nabla_\mu\Phi\nabla_\nu\Phi + \tfrac{1}{2}e^{4\Phi}g^{\mu\nu}\nabla_\mu K_a\nabla_\nu K_a + e^{-2\Phi}g^{\mu\lambda}g^{\nu\rho}F_{\mu\nu}F_{\lambda\rho} + K_a F_{\mu\nu}\tilde{F}^{\mu\nu} \tag{3.2}$$

where Σ is the spacetime region with metric g, K is the trace of the extrinsic curvature of the boundary $\partial\Sigma(r = \text{constant})$, K_0 is obtained by substituting into K the flat spacetime metric, h is the determinant of the metric induced on the boundary, and g is a formally Euclidean black hole metric, which can be obtained by setting $t \to i\tau$, $\varphi' \to i\varphi$ in Eq. (2.2) and expressed as

$$\begin{aligned} ds^2 = {} & \frac{\Sigma - a^2 \sin^2\theta}{\Delta} d\tau^2 + \frac{\Delta}{\Sigma} dr^2 + \Delta d\theta^2 \\ & - \frac{\sin^2\theta}{\Delta}\left[\left(r^2 + a^2 - 2Dr\right)^2 - \Sigma a^2 \sin^2\theta\right] d\varphi^2 \\ & + \frac{2a\sin^2\theta}{\Delta}\left[\left(r^2 + a^2 - 2Dr\right) - \Sigma\right] d\tau\, d\varphi, \end{aligned} \tag{3.3}$$

where τ has period $\beta = 2\pi/\kappa$, and κ is the surface gravity given by

$$\kappa = -\tfrac{1}{2}\left[\sqrt{\frac{-g^{rr}g_{\varphi'\varphi'}}{g_{tt}g_{\varphi'\varphi'} - g_{t\varphi'}^2}}\,\frac{\partial}{\partial r}\left(g_{tt} - \frac{g_{t\varphi'}^2}{g_{\varphi'\varphi'}}\right)\right]_{r=r_h} = \frac{r_+ - r_-}{2Mr_+}. \tag{3.4}$$

In order to get the expression of action I, let us start with the definition of K, the trace of the extrinistic curvature of the hypersurface with a spacelike unit normal vector n^μ:

$$K = h^{\mu\nu}\nabla_\mu n_\nu. \tag{3.5}$$

where

$$h_{\mu\nu} = g_{\mu\nu} - n_\mu n_\nu \tag{3.6}$$

is the induced metric on the hypersurface. For the stationary charged metric, setting $n_\mu = (0, \sqrt{g_{rr}}, 0, 0)$ as Ref. [8], we find that

$$\begin{aligned} K = {} & \frac{\sqrt{g_{rr}}}{2g_{rr}g_{\theta\theta}\left(g_{\tau\tau}g_{\varphi\varphi} - g_{\tau\varphi}^2\right)} \\ & \times\left\{\left(g_{\tau\tau}g_{\varphi\varphi} - g_{\tau\varphi}^2\right)\frac{\partial g_{\theta\theta}}{\partial r} + g_{\tau\tau}g_{\theta\theta}\frac{\partial g_{\varphi\varphi}}{\partial r} - g_{\tau\varphi}g_{\theta\theta}\frac{\partial g_{\tau\varphi}}{\partial r} + g_{\varphi\varphi}g_{\theta\theta}\frac{\partial g_{\tau\tau}}{\partial r}\right\}. \end{aligned} \tag{3.7}$$

$$\sqrt{h} = \sqrt{g_{\tau\tau}g_{\varphi\varphi} - g_{\tau}^2\varphi}\; g_{\theta\theta}. \tag{3.8}$$

Substituting (2.2)–(2.7), (3.3)–(3.8) into (3.1) and noting the position of the spherical cavity wall $\partial\Sigma$ at infinity, we obtain

$$I = \frac{\beta}{2}\left(M - \frac{Q^2}{2\omega M}\right). \tag{3.9}$$

In the next section we shall use this quantity to study thermodynamics of the stationary axisymmetric EMDA black hole.

4. Thermodynamics of stationary axisymmetric EMDA black hole

As usual in the study of the thermodynamics of a statistical system, we first introduce a thermodynamic potential function and a Helmoltz free energy, and then derive some thermodynamic quantities form them. In the presence of a set of conserved charge Q_i and their related potentials μ_i it is convenient to work in the grand canonical ensemble, where the fundamental object is the grand partition function

$$Z = \mathrm{Tr}\, e^{-\beta(H-\mu_i Q_i)}, \tag{4.1}$$

and the thermodynamic potential function [9]

$$W = E - TS - \mu_i Q_i - \Omega_{r_+} J, \tag{4.2}$$

where Ω_{r_+} and J are respectively the angular velocity and angular momentum of the stationary axisymmetric EMDA black hole, and Ω_{r_+} is given by $\Omega_{r_+} = a/(r_+^2 - 2Dr_+ + a^2) = J/2M^2 r_+$. The thermodynamic potential function is related to the grand partition function by [9]

$$e^{-\beta W} = Z. \tag{4.3}$$

As argued in [9], the partition function for the system can be defined by a Lagrangian path integral for the gravitational action coupled with matter fields,

$$Z = e^{-I}. \tag{4.4}$$

where I is the Euclidean action of the system, which is explicitly given by Eq. (3.9).

From Eqs. (4.3), (4.4) and (3.9) we get

$$W = -\frac{1}{\beta}\ln Z = \frac{I}{\beta} = \frac{1}{2}\left(M - \frac{Q^2}{2\omega M}\right). \tag{4.5}$$

The chemical potential [10] for the charge is given by

$$\mu_Q = -\left(\frac{\partial W}{\partial Q}\right)_\beta = \frac{Q}{2\omega M} = \frac{Qr^+}{\omega\left(r_+^2 + a^2 - 2Dr_+\right)}, \tag{4.6}$$

which is different from that in the Kerr–Newman black hole: $\mu_Q = Qr_+/r_+^2 + a^2$ [9].

Let us introduce the Helmoltz free energy [12]

$$F = E - TS = W + \mu_Q Q + \Omega_{r_+} J = \frac{1}{2}\left(M + \frac{Q^2}{2\omega M} + \frac{J^2}{M^2 r_+}\right). \tag{4.7}$$

The total classical entropy of the stationary axisymmetric EMDA black hole can be calculated from the Helmoltz free energy [12] to be

$$S = \beta^2\left(\frac{\partial F}{\partial \beta}\right)_Q = 2\pi M\left(M - \frac{Q^2}{2\omega M} + \sqrt{\left(M - \frac{Q^2}{2\omega M}\right)^2 - \frac{J^2}{M^2}}\right). \tag{4.8}$$

Noting the area of the event horizon:

$$A=\int\sqrt{g_{\theta\theta}g_{\varphi\varphi}}\;\mathrm{d}\theta\;\mathrm{d}\varphi=4\pi\left(r_+^2+a^2-2Dr_+\right)$$

$$=8\pi M\left(M-\frac{Q^2}{2\omega M}+\sqrt{\left(M-\frac{Q^2}{2\omega M}\right)^2-\frac{J^2}{M^2}}\right), \tag{4.9}$$

we have

$$S=\frac{A}{4}, \tag{4.10}$$

which shows that the classical Bekenstein–Hawking entropy of the stationary axisymmetric EMDA black hole is proportional to the area of the horizon as usual. We can show that the horizon area $A\neq 0$ in the case $J\neq 0$, but the horizon area A vanishes at $M=\sqrt{Q^2/2\omega}$ in the case $J=0$ (i.e., for a static black hole). The result in the case $J=0$ is similar to that of the static dilatonic black hole obtained by A. Ghosh and P. Mitra [4], By the method used in Ref. [13] we can prove that the total entropy never decreases in any physical process — the second law.

The contribution of the black hole to the energy E [14] is

$$E=\left(\frac{\partial\beta F}{\partial\beta}\right)_Q=M. \tag{4.11}$$

The temperature is

$$T=\left(\frac{\partial E}{\partial S}\right)_Q=\frac{1}{2\pi}\frac{r_+-r_-}{2Mr_+}=\frac{1}{2\pi}\frac{r_+-r_-}{r_+^2-2Dr_++a^2}=\frac{\kappa}{2\pi} \tag{4.12}$$

from which we obtain that the surface gravity of the stationary black hole (at equilibrium) is constant on the entire surface of the event horizon, i.e., the zeroth law. Using Eq. (4.12) we can prove the third law: it is impossible by any physical process to reduce κ to zero by a finite sequence of operations.

Differentiating (4.9) and using the above arguments we obtain

$$\mathrm{d}E=\frac{\kappa}{8\pi}\,\mathrm{d}A+\frac{Q^2}{2\omega M}\,\mathrm{d}Q+\frac{J}{2M^2r_+}\,\mathrm{d}J=T\,\mathrm{d}S+\mu_Q\,\mathrm{d}Q+Q_{r_+}\mathrm{d}J. \tag{4.13}$$

which is just an expression of the first law of thermodynamics stating that in an isolated system the total energy of the system is conserved. Eq. (4.13) has the same form as the classical first law of the Kerr–Newman black hole, but we should note that μ_Q, Ω_{r_+}, T and A have different forms.

The heat capacity C_{QJ} [13] with constant charge is then given as

$$C_{QJ}=-T\left(\frac{\partial^2F}{\partial^2T}\right)_{Q,J}=\frac{8MS^3T}{J^2-8S^3T^2}, \tag{4.14}$$

which differs from the Kerr–Newman black hole heat capacity [16] $C_{QJ} = 8MS^3T/J^2 + \frac{1}{4}Q^2 - 8S^3T^2$, but has the same form as the Kerr black hole $C_{QJ} = 8MS^3T/(J^2 - 8S^3T^2)$. It is obvious that heat capacity of the stationary axisymmetric EMDA black hole has singularities at $J^2 - 8S^3T^2 = 0$, i.e., $3M^8 - 8M^6Q^2 - 6M^4(J^2 - Q^4) - (J^2 - Q^4)^2 = 0$. At the same time, both F and S are continuous at the heat capacity's singularities. Thus, we have for this charged black hole second-order phase transitions just as for the Kerr–Newman black hole [15].

5. Conclusion and discussion

We have studied the thermodynamics of a stationary axisymmetric EMDA black hole by use of general statistical physics methods. We have first worked out the action and then introduced the partition function, thermodynamic potential function and Helmoltz free energy. Some thermodynamic quantities were then derived from them. Eqs. (4.10) and (4.11) show that the entropy and the energy have the same form as for the Kerr–Newman black hole, but Eqs. (4.12), (4.14) and (4.6) show that temperature, heat capacity and chemical potential have a different form. However, the Bardeen–Carter–Hawking laws of black hole thermodynamics [17] are valid for the stationary axisymmetric EMDA black hole. Since heat capacity of the stationary axisymmetric EMDA black hole is divergent but both F and S are continuous at some value of J and Q, the black hole possesses second-order phase transitions as does the Kerr–Newman black hole. Comparing Eq. (3.9) with Eq. (4.8) we obtained another interesting result, namely that action (the boundary at infinity) for the stationary axisymmetric EMDA black holes can be expressed as $S + \beta\Omega_{r_h}J$. This result is valid also for Kerr black hole [9] and Kerr–Newman black hole [9]. The result is also suitable for static black hole since static black hole is a special case of stationary black hole ($J = 0$). Examples include the Schwarzschild black hole [6], Reissner–Nordstrom black hole [9], Kallosh–Linde–Peet–Proeyen dilaton black hole [3] and black hole with global monopoles [18]. Since the thermodynamic potential function $W = E - TS - \Omega J - \mu_i Q_j$ is a formula for general stationary black holes [9] and we can obtain by means of Refs. [17,19] that for general stationary black holes $E = 2TS + 2\Omega_{r_h}J + \mu_i Q_j$, using the thermodynamic potential W and energy E given here and the relation $W = -(1/\beta)\ln Z = I/\beta$ we can prove that $I = S + \beta\Omega_{r_h}J$ is a universal relation for general stationary black holes in which the external material contribution to mass and angular momentum vanish. When $J = 0$, i.e., for static black holes, the action reduces to $I = S$, which is equivalent to the result obtained by R. Kallosh, T. Ortin and A. Peet [3].

Acknowledgement

This work was supported by the National Natural Science Foundation of China.

References

[1] R. Kallosh, A. Linde, T. Ortin, A. Peet and A. Van Proeyen, Phys. Rev. D 46 (1992) 5278;
G.W. Gibbons and K. Maeda, Nucl. Phys. B 298 (1988) 741;
R. Kallosh, D. Kastor, T. Ortin and T. Torma, Phys. Rev. D 50 (1994) 6374;
B. Jensen, U. Lindstrom and G.T. Horowitz, Phys. Rev. D 52 (1995) 3543;
D. Garfinkle, G.T. Horowitz and A. Strominger, Phys. Rev. D 43 (1991) 3140.

[2] A. Garcia, D. Galtsov and O. Kechkin, Phys. Rev. Lett. 74 (1995) 1276.

[3] R. Kallosh, T. Ortin and A. Peet, Phys. Rev. D 47 (1993) 5400.

[4] A. Ghosh and P. Mitra, Phys. Rev. Lett. 73 (1994) 2521.

[5] S.W. Hawking, *in* General relativity, ed. S.W. Hawking and W. Israel (Cambridge Univ. Press, Cambridge, England, 1979).

[6] J.W. York, Jr., Phys. Rev. D 33 (1986) 2092.

[7] B.F. Whiting and J.W. York, Jr., Phys. Rev. Lett. 61 (1988) 1336.

[8] P.C.W. Davies, L.H. Ford and D.N. Page, Phys. Rev. D 34 (1986) 1700.

[9] G.W. Gibbons and S.W. Hawking, Phys. Rev. D 15 (1977) 2752.

[10] H.W. Braden, J.D. Brown, B.F. Whiting and J.W. Yory, Jr., Phys. Rev. D 42 (1990) 3376.

[11] S. Coleman, J. Preskill and F. Wilczek, Quantum hair on black hole, Nucl. Phys. B 378 (1992) 175.

[12] L. Susskind and J. Uglum, Phys. Rev. D 50 (1994) 2700.

[13] C.O. Lousto, Nucl. Phys. B 410 (1993) 155.

[14] G. 't Hooft, Nucl. Phys. B 256 (1985) 727.

[15] Y.K. Lau, Phys. Lett. A 188 (1994) 245.

[16] P.C.W. Davies, Proc. R. Soc. London A 353 (1977) 499;
P. Hut, Mon. Not. R. Astron. Soc. 180 (1977) 379.

[17] J.M. Bardeen, B. Carter and S.W. Hawking, Commun. Math. Phys. 31 (1973) 161.

[18] H.W. Yu, Nucl. Phys. B 430 (1994) 427.

[19] L. Smarr, Phys. Rev. Lett. 30 (1972) 71;
B. Carter, *in* General relativity: A centenary survey, ed. S.W. Hawking and W. Israel (Cambridge Univ. Press, Cambridge, England, 1979);
J.L. Jing, Y. Wang and J.Y. Zhu, Phys. Lett. A 187 (1994) 31;
J. Bardeen, Astrophys. J. 162 (1970) 71.